USAMO 1996/2

Evan Chen

Twitch Solves ISL

Episode 81

Problem

For any nonempty set S of real numbers, let $\sigma(S)$ denote the sum of the elements of S. Given a set A of n positive integers, consider the collection of all distinct sums $\sigma(S)$ as S ranges over the nonempty subsets of A. Prove that this collection of sums can be partitioned into n classes so that in each class, the ratio of the largest sum to the smallest sum does not exceed 2.

Video

https://youtu.be/CkPmHY8MIYg

Solution

By induction on n with $n = 1$ being easy.

For the inductive step, assume

$$A = \{a_1 > a_2 > \dots > a_n\}.$$

Fix any index k with the property that

$$a_k > \frac{\sigma(A)}{2^k}$$

(which must exist since $\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^k} < 1$). Then

- We make k classes for the sums between $\frac{\sigma(A)}{2^k}$ and $\sigma(A)$; this handles every set which has any element in $\{a_1, \dots, a_k\}$.
- We make $n - k$ classes via induction hypothesis on $\{a_{k+1}, \dots, a_n\}$.

This solves the problem.