Korea 2021/2 Evan Chen

TWITCH SOLVES ISL

Episode 81

Problem

The incenter and A-excenter of $\triangle ABC$ is I and O. The foot from A, I to BC is D and E. The intersection of AD and EO is X. The circumcenter of $\triangle BXC$ is P. Show that the circumcircle of $\triangle BPC$ is tangent to the A-excircle if X is on the incircle of $\triangle ABC$.

Video

https://youtu.be/R7LQQU8NN9I

External Link

https://aops.com/community/p23632170

Solution

It's known that X coincides with the midpoints of \overline{AD} and also lies on line \overline{IF} .



Claim. (BXC) is tangent to the incircle and passes through the midpoint N of \overline{EO} .

Proof. Follows by 2002 G7.

Hence by homothety at X the line \overline{XIF} passes through P.

Claim. (BXNC) and the A-excircle are orthogonal.

Proof. It suffices to show BXCN is fixed under inversion around the A-excircle. To this end, we prove that

$$ON \cdot OX = OF^2.$$

Indeed, this follows from the similar isosceles triangles

$$\triangle ONF \sim \triangle OFX \sim \triangle EIX.$$

Hence it follows that inversion centered at P with radius PB = PC will fix the A-excircle. Since BC is tangent to the A-excircle at F, the inverse image of F is the desired tangency point.