Korea 2021/2

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Twitch Solves ISL

Episode 81

Problem

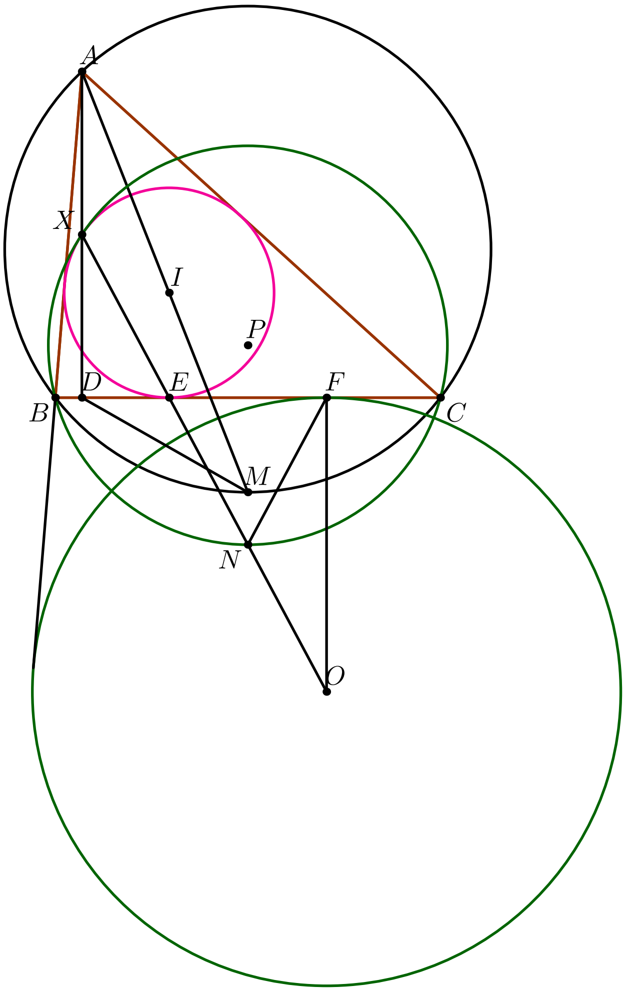
The incenter and A-excenter of $\triangle ABC$ is I and O. The foot from A, I to BC is D and E. The intersection of AD and EO is X. The circumcenter of $\triangle BXC$ is P. Show that the circumcircle of $\triangle BPC$ is tangent to the A-excircle if X is on the incircle of $\triangle ABC$.

Video

https://youtu.be/R7LQQU8NN9I

Solution

It's known that X coincides with the midpoints of $\overline{AD}$ and also lies on line $\overline{IF}$.



Claim. (BXC) is tangent to the incircle and passes through the midpoint N of $\overline{EO}$.

Proof. Follows by 2002 G7. □

Hence by homothety at X the line $\overline{XIF}$ passes through P.

Claim. $(BXNC)$ and the A-excircle are orthogonal.

Proof. It suffices to show $BXCN$ is fixed under inversion around the A-excircle. To this end, we prove that

$$ON \cdot OX = OF^2.$$

Indeed, this follows from the similar isosceles triangles

$$\triangle ONF \sim \triangle OFX \sim \triangle EIX.$$ □

Hence it follows that inversion centered at P with radius $PB = PC$ will fix the A-excircle. Since BC is tangent to the A-excircle at F, the inverse image of F is the desired tangency point.