# Korea 2021/2 

## Evan Chen

Twitch Solves ISL

Episode 81

## Problem

The incenter and $A$-excenter of $\triangle A B C$ is $I$ and $O$. The foot from $A, I$ to $B C$ is $D$ and $E$. The intersection of $A D$ and $E O$ is $X$. The circumcenter of $\triangle B X C$ is $P$. Show that the circumcircle of $\triangle B P C$ is tangent to the $A$-excircle if $X$ is on the incircle of $\triangle A B C$.

## Video

https://youtu.be/R7LQQU8NN9I

## External Link

https://aops.com/community/p23632170

## Solution

It's known that $X$ coincides with the midpoints of $\overline{A D}$ and also lies on line $\overline{I F}$.


Claim. $(B X C)$ is tangent to the incircle and passes through the midpoint $N$ of $\overline{E O}$.
Proof. Follows by 2002 G7.
Hence by homothety at $X$ the line $\overline{X I F}$ passes through $P$.
Claim. $(B X N C)$ and the $A$-excircle are orthogonal.
Proof. It suffices to show $B X C N$ is fixed under inversion around the $A$-excircle. To this end, we prove that

$$
O N \cdot O X=O F^{2}
$$

Indeed, this follows from the similar isosceles triangles

$$
\triangle O N F \sim \triangle O F X \sim \triangle E I X .
$$

Hence it follows that inversion centered at $P$ with radius $P B=P C$ will fix the $A$-excircle. Since $B C$ is tangent to the $A$-excircle at $F$, the inverse image of $F$ is the desired tangency point.

