Shortlist 2020 G7

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Twitch Solves ISL

Episode 80

Problem

Let P be a point on the circumcircle of acute triangle ABC. Let D, E, F be the reflections of P in the A-midline, B-midline, and C-midline. Let ω be the circumcircle of the triangle formed by the perpendicular bisectors of AD, BE, CF.

Show that the circumcircles of $\triangle ADP$, $\triangle BEP$, $\triangle CFP$, and ω share a common point.

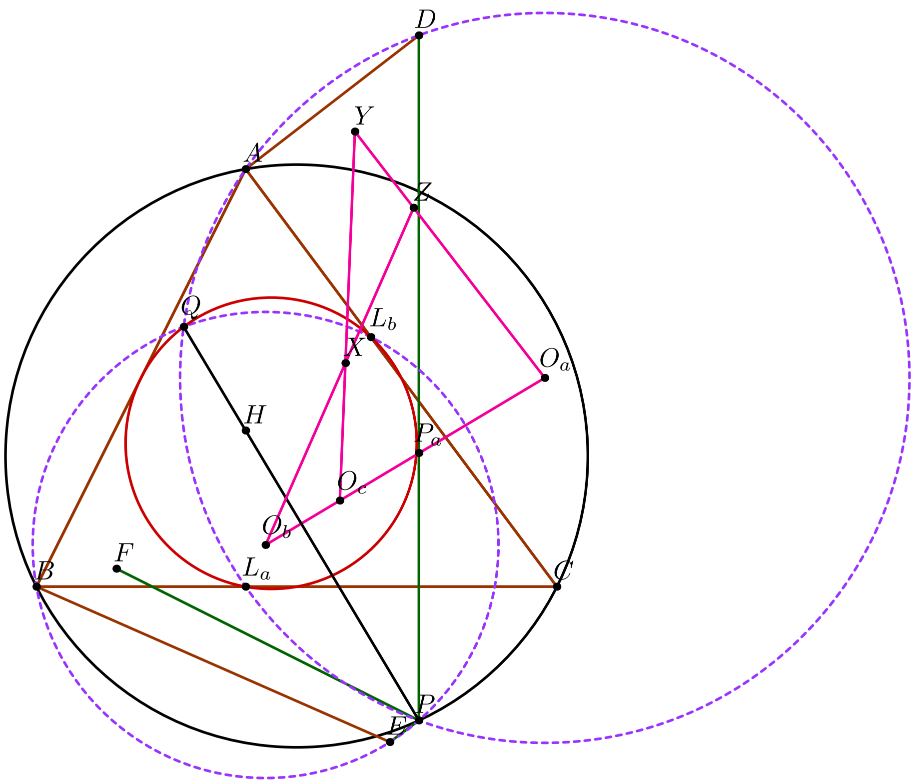
Video

https://youtu.be/0kODbd66TdE

Solution

The following solution was given by Ankan Bhattacharya.

Call the formed triangle XYZ. Also denote by O_a, O_b, O_c the centers of PAD, PBE, PCF. Let L_a, L_b, L_c be the altitude feet. Also, let ray PH meet the nine-point circle again at Q. We contend that Q is the desired point.



Claim. AQL_aPD is cyclic, and so on.

Proof. $ADPL_a$ is cyclic because it is an isosceles trapezoid, while AQL_aP is cyclic by power of a point from H. □

Claim. Q is the Miquel point of complete quadrilateral $XYZO_aO_bO_c$.

Proof. It is sufficient to show that Q lies on (O_aO_bZ). This is an angle chase:

$$\begin{aligned}
\measuredangle O_bQO_a &= \measuredangle O_bQP + \measuredangle PQO_a = (90^\circ - \measuredangle PL_bQ) + (90^\circ - \measuredangle QL_aP) \\
&= \measuredangle QL_bP + \measuredangle PL_aQ = \measuredangle L_aPL_b + \measuredangle L_bQL_a \\
&= \measuredangle L_aPL_b + 2\measuredangle ACB \\
\measuredangle O_bZO_a &= \measuredangle(BE, AD) = \arg(AD) - \arg(BE) \\
&= (2\arg(BC) - \arg(PL_a)) - (2\arg(AC) - \arg(PL_b)) \\
&= 2\measuredangle ACB - \measuredangle L_bPL_a
\end{aligned}$$

as needed. □