

IMO 2021/5

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TWITCH SOLVES ISL

Episode 79

Problem

Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favourite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the k th move, Jumpy swaps the positions of the two walnuts adjacent to walnut k .

Prove that there exists a value of k such that, on the k th move, Jumpy swaps some walnuts a and b such that $a < k < b$.

Video

<https://youtu.be/U52PMz8nJjI>

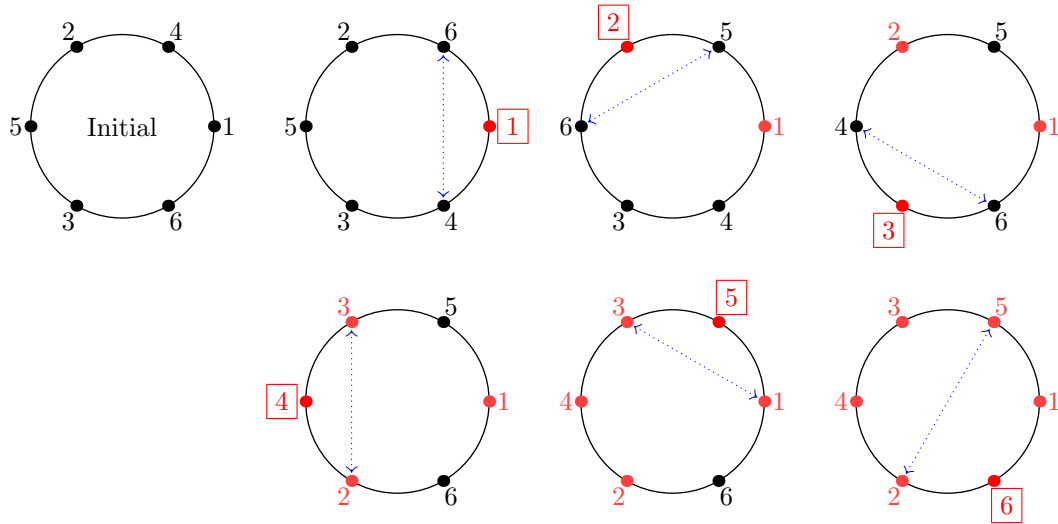
External Link

<https://aops.com/community/p22697921>

Solution

Assume for contradiction no such k exists.

This process takes exactly 2021 steps. Right after the k th move, we consider a situation where we color walnut k red as well, so at the k th step there are k ones. For brevity, a non-red walnut is called black. An example is illustrated below with 2021 replaced by 6.



Claim. At each step, the walnut that becomes red is between two non-red or two red walnuts.

Proof. By definition. □

On the other hand, if there are 2021 walnuts, one obtains a parity obstruction to this simplified process:

Claim. After the first step, there is always a consecutive block of black walnuts positive even length.

Proof. After the first step, there is a block of 2020 black walnuts.

Thereafter, note that a length 2 block of black walnuts can never be changed. Meanwhile for even lengths at least 4, if one places a red walnut inside it, the even length block splits into an odd length block and an even length block. □

Remark. The statement is true with 2021 replaced by any odd number, and false for any even number.

The motivation comes from the following rephrasing of the problem:

Start with all 0's and at each step change a 0 between two matching numbers from a 0 to a 1.

Although the coloring (or 0/1) argument may appear to lose information at first, I think it should be *equivalent* to the original process; the “extra” information comes down to the choice of which walnut to color red at each step.