# IMO 2021/4 

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Twitch Solves ISL
Episode 78

## Problem

Let $\Gamma$ be a circle with center $I$, and $A B C D$ a convex quadrilateral such that each of the segments $A B, B C, C D$ and $D A$ is tangent to $\Gamma$. Let $\Omega$ be the circumcircle of the triangle $A I C$. The extension of $B A$ beyond $A$ meets $\Omega$ at $X$, and the extension of $B C$ beyond $C$ meets $\Omega$ at $Z$. The extensions of $A D$ and $C D$ beyond $D$ meet $\Omega$ at $Y$ and $T$, respectively. Prove that

$$
A D+D T+T X+X A=C D+D Y+Y Z+Z C .
$$

## Video

https://youtu.be/MeXMi9NG2Rc

## External Link

https://aops.com/community/p22698001

## Solution

Let $P Q R S$ be the contact points of $\Gamma$ an $\overline{A B}, \overline{B C}, \overline{C D}, \overline{D A}$.


Claim. We have $\triangle I Q Z \cong \triangle I R T$. Similarly, $\triangle I P X \cong \triangle I S Y$.
Proof. By considering (CQIR) and (CITZ), there is a spiral similarity similarity mapping $\triangle I Q Z$ to $\triangle I R T$. Since $I Q=I R$, it is in fact a congruence.

This congruence essentially solves the problem. First, it implies:
Claim. $T X=Y Z$.
Proof. Because we saw $I X=I Y$ and $I T=I Z$.
Then, we can compute

$$
\begin{aligned}
A D+D T+X A & =A D+(R T-R D)+(X P-A P) \\
& =(A D-R D-A P)+R T+X P=R T+X P
\end{aligned}
$$

and

$$
\begin{aligned}
C D+D Y+Z C & =C D+(S Y-S D)+(Z Q-Q C) \\
& =(C D-S D-Q C)+S Y+Z Q=S Y+Z Q
\end{aligned}
$$

but $Z Q=R T$ and $X P=S Y$, as needed.

