IMO 2021/4 Evan Chen

TWITCH SOLVES ISL

Episode78

Problem

Let Γ be a circle with center I, and ABCD a convex quadrilateral such that each of the segments AB, BC, CD and DA is tangent to Γ . Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension of BCbeyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T, respectively. Prove that

AD + DT + TX + XA = CD + DY + YZ + ZC.

Video

https://youtu.be/MeXMi9NG2Rc

External Link

https://aops.com/community/p22698001

Solution

Let PQRS be the contact points of Γ an \overline{AB} , \overline{BC} , \overline{CD} , \overline{DA} .



Claim. We have $\triangle IQZ \cong \triangle IRT$. Similarly, $\triangle IPX \cong \triangle ISY$.

Proof. By considering (CQIR) and (CITZ), there is a spiral similarity similarity mapping $\triangle IQZ$ to $\triangle IRT$. Since IQ = IR, it is in fact a congruence.

This congruence essentially solves the problem. First, it implies:

Claim. TX = YZ.

Proof. Because we saw IX = IY and IT = IZ.

Then, we can compute

$$AD + DT + XA = AD + (RT - RD) + (XP - AP)$$
$$= (AD - RD - AP) + RT + XP = RT + XP$$

and

$$CD + DY + ZC = CD + (SY - SD) + (ZQ - QC)$$
$$= (CD - SD - QC) + SY + ZQ = SY + ZQ$$

but ZQ = RT and XP = SY, as needed.