IMO 2021/4

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Twitch Solves ISL

Episode 78

Problem

Let Γ be a circle with center I, and $ABCD$ a convex quadrilateral such that each of the segments AB, BC, CD and DA is tangent to Γ. Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T, respectively. Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC.$$

Video

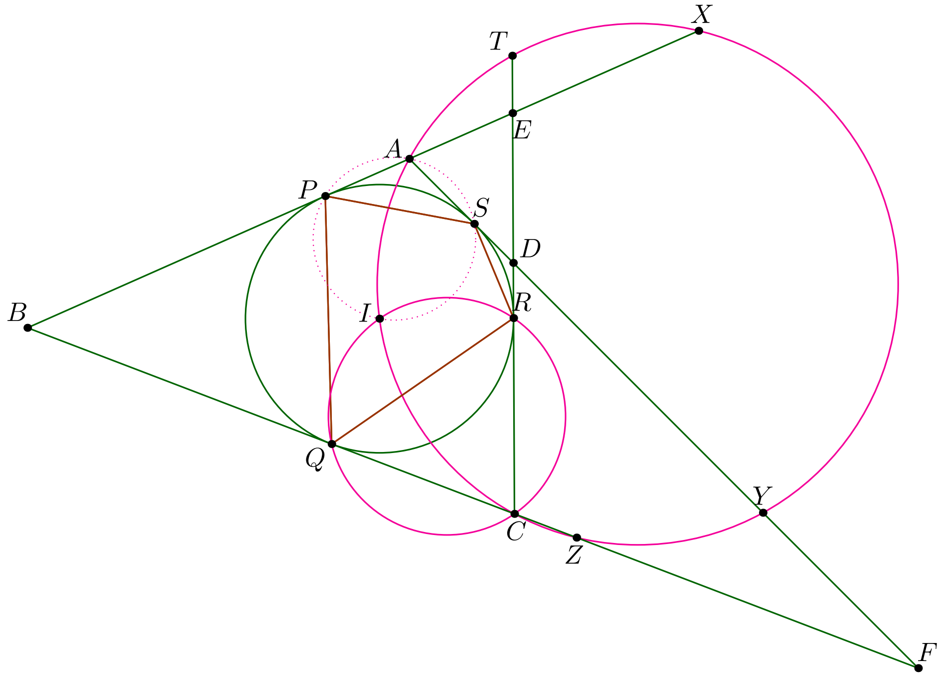
https://youtu.be/MeXMi9NG2Rc

External Link

https://aops.com/community/p22698001

Solution

Let $PQRS$ be the contact points of Γ an $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$.



Claim. We have $\triangle IQZ \cong \triangle IRT$. Similarly, $\triangle IPX \cong \triangle ISY$.

Proof. By considering $(CQIR)$ and $(CITZ)$, there is a spiral similarity similarity mapping $\triangle IQZ$ to $\triangle IRT$. Since $IQ = IR$, it is in fact a congruence. $\square$

This congruence essentially solves the problem. First, it implies:

Claim. $TX = YZ$.

Proof. Because we saw $IX = IY$ and $IT = IZ$. $\square$

Then, we can compute

$$\begin{aligned} AD + DT + XA &= AD + (RT - RD) + (XP - AP) \\ &= (AD - RD - AP) + RT + XP = RT + XP \end{aligned}$$

and

$$\begin{aligned} CD + DY + ZC &= CD + (SY - SD) + (ZQ - QC) \\ &= (CD - SD - QC) + SY + ZQ = SY + ZQ \end{aligned}$$

but $ZQ = RT$ and $XP = SY$, as needed.