Shortlist 2013 C4

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Twitch Solves ISL

Episode 77

Problem

Let n be a positive integer, and let A be a subset of $\{1, \dots, n\}$. An A-partition of n into k parts is a representation of n as a sum $n = a_1 + \cdots + a_k$, where the parts $a_1, \dots, a_k$ belong to A and are not necessarily distinct. The number of different parts in such a partition is the number of (distinct) elements in the set $\{a_1, a_2, \dots, a_k\}$. We say that an A-partition of n into k parts is optimal if there is no A-partition of n into r parts with $r < k$. Prove that any optimal A-partition of n contains at most $\sqrt[3]{6n}$ different parts.

Video

https://youtu.be/wWVB29XWuU0

External Link

https://aops.com/community/p3543403

Solution

Suppose we have an optimal partition

$$n = e_1 \cdot a_1 + \cdots + e_m \cdot a_m.$$

In order for this to be optimal, it follows that for any subsets I and J of the index set $\{1, \dots, m\}$ if $|I| \neq |J|$ then $\sum_{i \in I} a_i \neq \sum_{j \in J} a_j$. Otherwise, we could replace one sum with the other in our optimal partition and get one with fewer total parts.

This will be enough.

Claim. There are at least $k(m-k)+1$ possible values of $\sum_{i \in I} a_i$ across index sets $I \subseteq \{1, \dots, m\}$ of cardinality k.

Proof. Sort $a_1 < \cdots < a_m$. Start with

$$\begin{aligned} a_1 + \cdots + a_k &< a_1 + \cdots + a_{k-1} + a_{k+1} \\ &< a_1 + \cdots + a_{k-1} + a_{k+2} \\ &< \cdots \\ &< a_1 + \cdots + a_{k-1} + a_n \end{aligned}$$

which one can visualize as "moving a_k to a_n". Then move a_{k-1} to a_{n-1}, and so on. $\square$

Vary k. The sums we get must all be different between layers, and bounded by n. So we have

$$\begin{aligned} n + 1 &\geq \sum_{k=0}^{m} [k(m-k)+1] \\ &= m \cdot \frac{m(m+1)}{2} - \frac{m(m+1)(2m+1)}{6} + m \\ &= \frac{m(m+1)}{2} \cdot \frac{m-1}{3} + m \\ &= \frac{m(m^2-1)}{6} + m \geq \frac{m^3}{6} + 1 \end{aligned}$$

so $m \leq \sqrt[3]{6n}$.