Shortlist 2007 N1

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Episode 77

Problem

Find all pairs of positive integers (a, b) such that $7^a - 3^b$ divides $a^4 + b^2$.

Video

https://youtu.be/MjxGLjTWZdM

External Link

https://aops.com/community/p1106459

Solution

The only answer is $(a, b) = (2, 4)$, which works as $7^2 - 3^4 = -32$ divides $2^4 + 4^2 = 32$.

To show this is the only solution, we note the following:

- The left-hand side is always even, which implies a and b have the same parity.
- This in turn implies the left-hand side is 0 mod 4.
- Hence $a^4 + b^2 \equiv 0 \pmod 4$ which means a and b are even.
- This means that $7^a - 3^b \equiv 0 \pmod 8$, which further means $b \equiv 0 \pmod 4$.

This means we may let $a = 2x$ and $b = 2y$ with y even to obtain

$$(7^x - 3^y)(7^x + 3^y) \mid (2x)^4 + (2y)^2.$$

To reduce to finite casework we contend:

Claim. $2 \cdot 3^a > 16a^4 + 4a^2$ once $a \geq 10$.

Proof. $3^{10} = 243^2 = 2 \cdot 59049 = 177147 > 160400$. □

Hence we are done if $\max(x, y) \geq 10$. Next

Claim. $7^x > 16x^4 + 4 \cdot 10^2$ for $x \geq 6$

Proof. $7^6 = 343^2 > 100000 > 16 \cdot 6^4 + 400$. □

So we may assume $x \leq 5$ as well.

This gives us the following cases. In all situations, note that $(2 \cdot 5)^4 + (2 \cdot 10)^2 = 10400$ and consequently if $(7^x - 3^y)(7^x + 3^y) > 10400$ there is nothing to verify. These cases are marked as "big" in the table.

		7	49	343	2401	16807
		$x = 1$	$x = 2$	$x = 3$	$x = 4$	$x = 5$
9	$y = 2$	OK	$40 \cdot 58 \nmid 272$	big	big	big
81	$y = 4$	$74 \cdot 88 \nmid 48$	$32 \cdot 130 \nmid 320$	big	big	big
729	$y = 6$	big	big	big	big	big
6561	$y = 8$	big	big	big	big	big