# Shortlist 1999 C6

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Twitch Solves ISL

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## Problem

Suppose that every integer has been given one of the colours red, blue, green or yellow. Let $x$ and $y$ be odd integers so that $|x| \neq |y|$. Show that there are two integers of the same colour whose difference has one of the following values: $x, y, x+y$ or $x-y$.

## Video

https://youtu.be/iznvJAYuUqo

## External Link

https://aops.com/community/p131878

## Solution

Assume for contradiction a coloring

$$\chi \colon \mathbb{Z} \to S \coloneqq \{\text{red}, \text{green}, \text{blue}, \text{yellow}\}$$

existed violating the conclusion. Then, we construct a coloring of $\widehat{\chi} \colon \mathbb{Z}^2 \to S$ by

$$\widehat{\chi}\,(a, b) = \chi(x \cdot a + y \cdot b).$$

**Claim.** $\widehat{\chi}$ assigns different colors to $(a, b)$, $(a, b+1)$, $(a+1, b)$, $(a+1, b+1)$.

*Proof.* By definition. □

However colorings $\widehat{\chi} \colon \mathbb{Z}^2 \to S$ satisfying the claim are actually straightforward to describe completely. Once such a description is given, one can directly check it can't be the lift of a $\chi$ as described.