ELMO 2010/3

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Twitch Solves ISL

Episode 75

Problem

Let $n > 1$ be a positive integer. A 2-dimensional grid, infinite in all directions, is given. Each 1 by 1 square in a given n by n square has a counter on it. A move consists of taking n adjacent counters in a row or column and sliding them each by one space along that row or column. A returning sequence is a finite sequence of moves such that all counters again fill the original n by n square at the end of the sequence.

(a) Assume that all counters are distinguishable except two, which are indistinguishable from each other. Prove that any distinguishable arrangement of counters in the n by n square can be reached by a returning sequence.

(b) Assume all counters are distinguishable. Prove that there is no returning sequence that switches two counters and returns the rest to their original positions.

Video

https://youtu.be/rGoMTlJwq-I

Solution

Part (b) is good motivation for (a), so we do that first. We claim that we can only obtain even permutations. The proof takes two observations:

- On the one hand, consider $\sum(x+y)$ across all counters at each step of the operation. It increases by $+n$ or $-n$ at every step. So any returning sequence must have an even number of operations.
- On the other hand, at any time, we can consider the counter labeling as a permutation on $(\mathbb{Z}/n\mathbb{Z})^2$ (at any point, for any ordered pair of residues, exactly one counter occupies that residue). Then each operation is an n-cycle.

Hence, a returning sequence is a composition of an even number of n-cycles; consequently, it follows that each permutation is obtained has even *parity*. Therefore, it's impossible to achieve (b).

On the other hand, towards (a), we show any even permutation is achievable.

Claim. We can permute any "right triangle," i.e.

$$\begin{bmatrix} \ddots & \dots & \dots & \iddots \\ \vdots & A & B & \vdots \\ \vdots & C & \bullet & \vdots \\ \iddots & \dots & \dots & \ddots \end{bmatrix} \longrightarrow \begin{bmatrix} \ddots & \dots & \dots & \iddots \\ \vdots & B & C & \vdots \\ \vdots & A & \bullet & \vdots \\ \iddots & \dots & \dots & \ddots \end{bmatrix} \tag{1}$$

Proof. Push the row with A and B left, push the column with C and A up, push the same row right, push the same column down. $\square$

By composing these, we also get that

$$\begin{bmatrix} \ddots & \dots & \dots & \iddots \\ \vdots & A & B & \vdots \\ \vdots & C & D & \vdots \\ \iddots & \dots & \dots & \ddots \end{bmatrix} \longrightarrow \begin{bmatrix} \ddots & \dots & \dots & \iddots \\ \vdots & B & A & \vdots \\ \vdots & D & C & \vdots \\ \iddots & \dots & \dots & \ddots \end{bmatrix} \tag{2}$$

and

$$\begin{bmatrix} \ddots & \dots & \dots & \dots & \iddots \\ \vdots & A & B & C & \vdots \\ \vdots & \bullet & D & \bullet \vdots & \\ \iddots & \dots & \dots & \dots & \ddots \end{bmatrix} \longrightarrow \begin{bmatrix} \ddots & \dots & \dots & \dots & \iddots \\ \vdots & B & C & A & \vdots \\ \vdots & \bullet & D & \bullet \vdots & \\ \iddots & \dots & \dots & \dots & \ddots \end{bmatrix} \tag{3}$$

are both possible.

To finish, we number the counters in the following way:

$$\begin{bmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 2n & 2n-1 & 2n-2 & \dots & n+2 & n+1 \\ 2n+1 & 2n+2 & 2n+3 & \dots & 3n-1 & 3n \\ 4n & 4n-1 & 4n-2 & \dots & 3n+2 & 3n+1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Observe that any two consecutive numbers are consecutive. But we have the following fact:

Lemma (Bubble sort)**.** Let N be any integer. Any permutation on $\{1, 2, \dots, N\}$ is generated by the $N - 1$ *adjacent transpositions* $(1\ 2)$, $(2\ 3)$, ..., $(N-1\ N)$.

Proof. This is a standard fact, most easily proved by induction on N. □

It follows an *even* permutation on $\{1, 2, \dots, N\}$ is an even composition of adjacent transpositions. But:

Claim. Any two adjacent transpositions for our n^2 counters can be obtained as the product of operations of the form (1), (2), (3).

Proof. By cases:

- If both transpositions are horizontal, start with any operation of the form (2) and then "move" the two transpositions in place by using (2) and (3) repeatedly.
- Same for two vertical.
- If one horizontal and one vertical, then start with (1) and use the same proof.

□

This completes the proof.