# ELMO 2010/3 Evan Chen

TWITCH SOLVES ISL

Episode 75

#### Problem

Let n > 1 be a positive integer. A 2-dimensional grid, infinite in all directions, is given. Each 1 by 1 square in a given n by n square has a counter on it. A move consists of taking n adjacent counters in a row or column and sliding them each by one space along that row or column. A returning sequence is a finite sequence of moves such that all counters again fill the original n by n square at the end of the sequence.

- (a) Assume that all counters are distinguishable except two, which are indistinguishable from each other. Prove that any distinguishable arrangement of counters in the n by n square can be reached by a returning sequence.
- (b) Assume all counters are distinguishable. Prove that there is no returning sequence that switches two counters and returns the rest to their original positions.

## Video

https://youtu.be/rGoMTlJwq-I

## **External Link**

https://aops.com/community/p2731457

#### Solution

Part (b) is good motivation for (a), so we do that first. We claim that we can only obtain even permutations. The proof takes two observations:

- On the one hand, consider  $\sum (x+y)$  across all counters at each step of the operation. It increases by +n or -n at every step. So any returning sequence must have an even number of operations.
- On the other hand, at any time, we can consider the counter labeling as a permutation on  $(\mathbb{Z}/n\mathbb{Z})^2$  (at any point, for any ordered pair of residues, exactly one counter occupies that residue). Then each operation is an *n*-cycle.

Hence, a returning sequence is a composition of an even number of *n*-cycles; consequently, it follows that each permutation is obtained has even *parity*. Therefore, it's impossible to achieve (b).

On the other hand, towards (a), we show any even permutation is achievable.

Claim. We can permute any "right triangle," i.e.

$$\begin{bmatrix} \ddots & \dots & \ddots & \ddots \\ \vdots & A & B & \vdots \\ \vdots & C & \bullet & \vdots \\ \ddots & \dots & \ddots \end{bmatrix} \longrightarrow \begin{bmatrix} \ddots & \dots & \ddots & \ddots \\ \vdots & B & C & \vdots \\ \vdots & A & \bullet & \vdots \\ \ddots & \dots & \ddots \end{bmatrix}$$
(1)

*Proof.* Push the row with A and B left, push the column with C and A up, push the same row right, push the same column down.  $\Box$ 

By composing these, we also get that

$$\begin{bmatrix} \ddots & \cdots & \cdots & \ddots \\ \vdots & A & B & \vdots \\ \vdots & C & D & \vdots \\ \vdots & & \cdots & \ddots \end{bmatrix} \longrightarrow \begin{bmatrix} \ddots & \cdots & \cdots & \ddots \\ \vdots & B & A & \vdots \\ \vdots & D & C & \vdots \\ \vdots & & \cdots & \ddots \end{bmatrix}$$
(2)

and

$$\begin{bmatrix} \ddots & \dots & \dots & \ddots \\ \vdots & A & B & C & \vdots \\ \vdots & \bullet & D & \bullet \vdots \\ \ddots & \dots & \dots & \ddots \end{bmatrix} \longrightarrow \begin{bmatrix} \ddots & \dots & \dots & \ddots \\ \vdots & B & C & A & \vdots \\ \vdots & \bullet & D & \bullet \vdots \\ \vdots & \ddots & \dots & \dots & \ddots \end{bmatrix}$$
(3)

are both possible.

To finish, we number the counters in the following way:

$$\begin{bmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 2n & 2n-1 & 2n-2 & \dots & n+2 & n+1 \\ 2n+1 & 2n+2 & 2n+3 & \dots & 3n-1 & 3n \\ 4n & 4n-1 & 4n-2 & \dots & 3n+2 & 3n+1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Observe that any two consecutive numbers are consecutive. But we have the following fact:

**Lemma** (Bubble sort). Let N be any integer. Any permutation on  $\{1, 2, ..., N\}$  is generated by the N - 1 adjacent transpositions  $(1 \ 2), (2 \ 3), ..., (N - 1 \ N)$ .

*Proof.* This is a standard fact, most easily proved by induction on N.

It follows an *even* permutation on  $\{1, 2, ..., N\}$  is an even composition of adjacent transpositions. But:

**Claim.** Any two adjacent transpositions for our  $n^2$  counters can be obtained as the product of operations of the form (1), (2), (3).

*Proof.* By cases:

- If both transpositions are horizontal, start with any operation of the form (2) and then "move" the two transpositions in place by using (2) and (3) repeatedly.
- Same for two vertical.
- If one horizontal and one vertical, then start with (1) and use the same proof.

This completes the proof.