Brazil 2020/5

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Twitch Solves ISL

Episode 74

Problem

Let n and k be positive integers with $k \leq n$. In a group of n people, each one or always speak the truth or always lies. Arnaldo can ask questions for any of these people provided these questions are of the type: "In set A, what is the parity of people who speak truthfully?", where A is a subset of k people. (The answer is either "even" or "odd").

Arnaldo wants to figure out which people are liars. For each pair (n, k), determine the minimum number of questions needed, or prove that this task is impossible.

Video

https://youtu.be/2ku744eaxYA

Solution

The answer is that the task is possible exactly when k is even in which case exactly n questions are needed.

Treat each person as an element in $\mathbb{F}_2$, where 1 for truth and 0 for liar.

If you ask person p about the set A, their response is

$$(p+1) + \sum_{a \in A} \pmod 2.$$

So in other words, a query amounts to sampling a set of either $k-1$ elements ($p \in A$) or $k+1$ elements ($p \notin A$), and taking the sum.

Now, if k is odd, the task is impossible, because replacing every $x \mapsto x+1$ changes no responses.

On the other hand, when k is even, the following n queries suffice:

- Query $(x_1 + \cdots + x_k) - x_i$ for $i = 1, \ldots, k$ By summing, one gets the value of $(k-1)(x_1 + \cdots + x_k)$, and hence knows x_i for $1 \le i \le k$.
- Query $(x_1 + \cdots + x_{k-2}) + x_i$ for $i = k+1, \ldots, n$. This gets x_i for $i \ge n$.

Moreover, at least n queries are necessary because there are 2^n possible final answers for Arnaldo, and each query has two possible responses.