# Brazil 2020/5 

## Evan Chen

## Twitch Solves ISL

Episode 74

## Problem

Let $n$ and $k$ be positive integers with $k \leq n$. In a group of $n$ people, each one or always speak the truth or always lies. Arnaldo can ask questions for any of these people provided these questions are of the type: "In set $A$, what is the parity of people who speak truthfully?", where $A$ is a subset of $k$ people. (The answer is either "even" or "odd").

Arnaldo wants to figure out which people are liars. For each pair $(n, k)$, determine the minimum number of questions needed, or prove that this task is impossible.

## Video

https://youtu.be/2ku744eaxYA

## External Link

https://aops.com/community/p21012476

## Solution

The answer is that the task is possible exactly when $k$ is even in which case exactly $n$ questions are needed.

Treat each person as an element in $\mathbb{F}_{2}$, where 1 for truth and 0 for liar.
If you ask person $p$ about the set $A$, their response is

$$
(p+1)+\sum_{a \in A}(\bmod 2) .
$$

So in other words, a query amounts to sampling a set of either $k-1$ elements $(p \in A)$ or $k+1$ elements $(p \notin A)$, and taking the sum.

Now, if $k$ is odd, the task is impossible, because replacing every $x \mapsto x+1$ changes no responses.

On the other hand, when $k$ is even, the following $n$ queries suffice:

- Query $\left(x_{1}+\cdots+x_{k}\right)-x_{i}$ for $i=1, \ldots, k$ By summing, one gets the value of $(k-1)\left(x_{1}+\cdots+x_{k}\right)$, and hence knows $x_{i}$ for $1 \leq i \leq k$.
- Query $\left(x_{1}+\cdots+x_{k-2}\right)+x_{i}$ for $i=k+1, \ldots, n$. This gets $x_{i}$ for $i \geq n$.

Moreover, at least $n$ queries are necessary because there are $2^{n}$ possible final answers for Arnaldo, and each query has two possible responses.

