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Twitch Solves ISL

Episode 73

Problem

Let G be a finite connected graph. Suppose that, if we choose any odd cycle in G and delete all its edges, the resulting graph is not connected. Prove that we color each vertex of G with one of four colors such that no two adjacent vertices are the same color.

Video

https://youtu.be/uDVCCdQLt6s

Solution

Let T be a spanning tree, and let H be the subgraph (not induced) of remaining edges.

- Trees are bipartite, so T has a two-coloring.
- By condition, H has no odd cycles (since deletion of an odd cycle should disconnect G). Hence H is bipartite, and has a two-coloring.

Then we can four-color G using the ordered pair of two-colorings.