Twitch 072.1

Evan Chen

Twitch Solves ISL

Episode 72

Problem

Each point of a three-dimensional space is colored with one of two colors such that whenever an isosceles triangle ABC with $AB = AC$ has vertices of the same color c it follows that the midpoint of BC also is colored with c. Prove that there exists a perpendicular square prism with all vertices of equal color.

Video

https://youtu.be/7Qjr5CwzQDU

Solution

Assume for contradiction that this is not the case, and call the colors red and green. We proceed in several steps.

Claim. The midpoint of any two points of the same color is the same color.

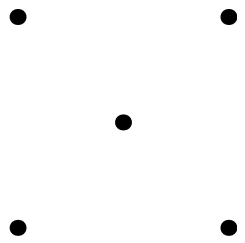
Proof. Assume for contradiction we found two green points B and C whose midpoint is red. Then every point on the perpendicular bisecting plane $\mathcal{P}$ of B and C is necessarily also red.

Now, take any red point X not on $\mathcal{P}$ (it clearly exists or we'd be done). Then every point on the plane halfway between X and $\mathcal{P}$ red would be red too; since if the midpoint of XP was green for $P \in \mathcal{P}$, then we can complete it to a red isosceles triangle XPQ for some $Q \in \mathcal{P}$.

Since we have two parallel entirely red planes, contradiction. □

So, assume henceforth the claim.

Claim. In any two-coloring of the following set, there is a monochromatic isosceles right triangle:



Proof. Exhaustive check. (Note that if two opposite vertices are the same color, the midpoint shares that color too.) □

Take 33 parallel stacks of that set. Then you can find two whose colorings are the same. On the other hand, given a monochromatic right isosceles triangle, taking all three midpoints gives a monochromatic square. The end.