# Twitch 072.1 

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Twitch Solves ISL
Episode 72

## Problem

Each point of a three-dimensional space is colored with one of two colors such that whenever an isosceles triangle $A B C$ with $A B=A C$ has vertices of the same color $c$ it follows that the midpoint of $B C$ also is colored with $c$. Prove that there exists a perpendicular square prism with all vertices of equal color.

## Video

https://youtu.be/7Qjr5CwzQDU

## Solution

Assume for contradiction that this is not the case, and call the colors red and green. We proceed in several steps.

Claim. The midpoint of any two points of the same color is the same color.
Proof. Assume for contradiction we found two green points $B$ and $C$ whose midpoint is red. Then every point on the perpendicular bisecting plane $\mathcal{P}$ of $B$ and $C$ is necessarily also red.

Now, take any red point $X$ not on $\mathcal{P}$ (it clearly exists or we'd be done). Then every point on the plane halfway between $X$ and $\mathcal{P}$ red would be red too; since if the midpoint of $X P$ was green for $P \in \mathcal{P}$, then we can complete it to a red isosceles triangle $X P Q$ for some $Q \in \mathcal{P}$.

Since we have two parallel entirely red planes, contradiction.
So, assume henceforth the claim.
Claim. In any two-coloring of the following set, there is a monochromatic isosceles right triangle:

Proof. Exhaustive check. (Note that if two opposite vertices are the same color, the midpoint shares that color too.)

Take 33 parallel stacks of that set. Then you can find two whose colorings are the same. On the other hand, given a monochromatic right isosceles triangle, taking all three midpoints gives a monochromatic square. The end.

