ToT Fall 2011 S-A7

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Twitch Solves ISL

Episode 72

Problem

Suppose 100 red points divide a blue circle into 100 arcs such that their lengths are all positive integers from 1 to 100 in an arbitrary order. Prove that there exist two perpendicular chords with red endpoints.

Video

https://youtu.be/DgsH-Nn2f0o

External Link

https://aops.com/community/p14404365

Solution

We will assume no two red points are diametrically opposite, otherwise, the problem is trivial.

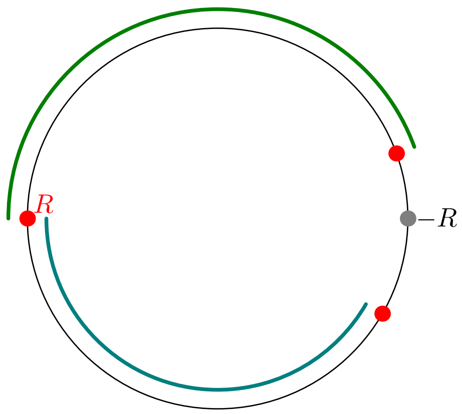
Claim. If there exist two intervals which have sum of lengths 2525, and it is not the case that

- the intervals share an endpoint; and
- one interval contains the other

then the problem is solved.

Proof. If two of the endpoints coincide, but the intervals are disjoint, then we have diametrically opposite red points, a contradiction. So assume we have four distinct red points in play. If the intervals are disjoint, then we get two chords which are perpendicular inside the circle; if they are nested, we get two chords perpendicular outside the circle. $\square$

For every red point R, we define its two *attached arcs* to be the two maximal-by-inclusion arcs with endpoint R and total length at most 2525. See figure below with two attached arcs drawn.



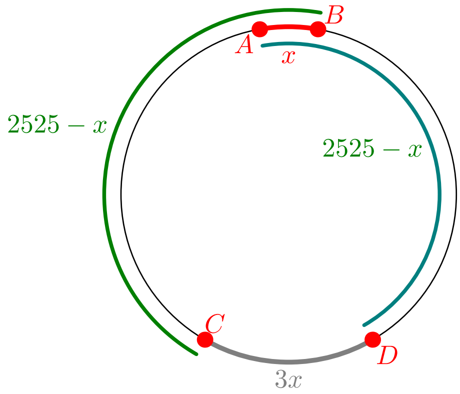
Then, of these two, we pick the one with longest length and call it the *main arc* of R. A main arc Γ always has length $2525 - x$ for some $0 < x \le 50$. We refer to the given arc of length x as the *killer* of Γ. Now, the problem is solved unless we are in a situation where for every main arc Γ, its killer is contained in Γ and shares an endpoint with it.

Claim. If there exists any main arc with $x = 50$, we are done.

Proof. It means the two attached arcs both have length exactly 2475, and that the segment of length 100 is split exactly by the antipode of R. So the segment of length 50 can't be killers of both. $\square$

Claim. If any arc of length $x < 50$ is the killer of two different main arcs, we are done.

Proof. If it was the killer of two attached arcs, then we have the following picture.



Since $2x < 100$, we can consider the arc of length $2x$. Both arcs AC and BD have length $2525 - 2x$. The arc of length $2x$ can't share an endpoint with both at once. $\square$

Finally, note that there are at least 50 main arcs (each choice of R generates one, and each main arc appears at most twice this way), but only 49 possible killers. So this solves the problem.