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TWITCH SOLVES ISL

Episode 72

Problem

For which positive integers n is the polynomial

$$p(x) = 1 + x^n + x^{2n}$$

reducible over the integers?

Video

<https://youtu.be/drrHDLWR6bw>

Solution

The answer is only n a power of 3 (including 1). Indeed, letting Φ_\bullet denote the cyclotomic polynomial, we have the decomposition of p into irreducibles is exactly given by

$$p(x) = \frac{x^{3n} - 1}{x^n - 1} = \frac{\prod_{d|3n} \Phi_d(n)}{\prod_{d|n} \Phi_d(n)} = \prod_{\substack{d|3n \\ d \nmid n}} \Phi_d(n).$$

Hence the answer is those n for which there is only one d satisfying $d \mid 3n$, $d \nmid n$. This is exactly the powers of 3.