Shortlist 2012 N3

Evan Chen

Twitch Solves ISL

Episode 70

Problem

Determine all integers $m \geq 2$ such that every n with $\frac{m}{3} \leq n \leq \frac{m}{2}$ divides the binomial coefficient $\binom{n}{m-2n}$.

Video

https://youtu.be/zVmGnDNw6LE

Solution

The answer is m prime. The canonical way to think of this is:

Claim. The number m works if and only if $\gcd(n, m) = 1$ for all $n \in [m/3, m/2]$.

Proof. Fix m, and choose any prime p. The divisibility is rewritten as

$$\nu_p\left((n-1)!\right) \geq \nu_p\left((m-2n)!\right) + \left(\nu_p(2n)!\right)$$

Define

$$f(n,q) = \left\lfloor \frac{n-1}{q} \right\rfloor - \left\lfloor \frac{m-2n}{q} \right\rfloor - \left\lfloor \frac{3n-m-1}{q} \right\rfloor.$$

Then we have

$$\nu_p\left(\binom{n}{m-2n}\right) = \sum_{e \geq 1} f(n, p^e).$$

We want this sum to be nonnegative for any prime p.

The following properties are true:

- $f(n,q) \geq 0$ if $q \nmid n$, since then $\left\lfloor \frac{n-1}{q} \right\rfloor = \left\lfloor \frac{n}{q} \right\rfloor$. So there are no issues when $p \nmid n$.
- Now suppose q is a prime power of a prime p dividing n. We may explicitly compute

$$\begin{aligned} f(qk,q) &= (k-1)) - \left(\left\lfloor \frac{m-2(qk)}{q} \right\rfloor + \left\lfloor \frac{3(qk)-m-1}{q} \right\rfloor \right) \\ &= -1 - \left\lfloor \frac{m}{q} \right\rfloor - \left\lfloor \frac{-(m+1)}{q} \right\rfloor \\ &= \begin{cases} -1 & q \mid m \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

  In that case, it follows that $\nu_p(\binom{n}{m-2n})$ is the sum of some non-positive terms, at least one of which is -1 (namely the first term $f(n,p^1) = -1$.) So m fails.

In conclusion, m works if and only n never has a common prime factor with m, as desired. □