

Junior Balkan 2013/3

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TWITCH SOLVES ISL

Episode 70

Problem

Show that

$$\left(a + 2b + \frac{2}{a+1}\right) \left(b + 2a + \frac{2}{b+1}\right) \geq 16$$

for all positive real numbers a and b such that $ab \geq 1$.

Video

<https://youtu.be/DQ0-cuYJ-mQ>

Solution

It's enough to prove

$$\left((b+1) + \frac{2}{a+1} + 1 \right) \left(\frac{2}{b+1} + (a+1) + 1 \right) \geq 16.$$

Let $x = a + 1$ and $y = b + 1$, so $xy \geq ab + a + b + 1 \geq ab + 2\sqrt{ab} + 1 = 4$. Then

$$\begin{aligned} \left(y + \frac{2}{x} + 1 \right) \left(\frac{2}{y} + x + 1 \right) &= \frac{4}{xy} + xy + 1 + 2 + 2 + x + y + \frac{2}{x} + \frac{2}{y} \\ &= 5 + \frac{4}{xy} + xy + x + y + \frac{2}{x} + \frac{2}{y} \\ &= 5 + 4 \cdot \frac{1}{xy} + xy + x + y + 2 \cdot \frac{1}{x} + 2 \cdot \frac{1}{y} \\ &\geq 5 + 4 \cdot \frac{1}{xy} + xy + 2 \left(\sqrt{xy} + \frac{2}{\sqrt{xy}} \right) \end{aligned}$$

On the other hand, it's easy to prove the following two assertions by clearing the denominator:

- For any $t \geq 4$, we have $t + \frac{4}{t} \geq 5$.
- For any $t \geq 2$, we have $t + \frac{2}{t} \geq 3$.

Applying this finishes the problem.