

Bulgaria 2018/3

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TWITCH SOLVES ISL

Episode 69

Problem

Prove that

$$\left(\frac{6}{5}\right)^{\sqrt{3}} > \left(\frac{5}{4}\right)^{\sqrt{2}}.$$

Video

<https://youtu.be/xEy1PKfyDaM>

External Link

<https://aops.com/community/p10202343>

Solution

Because $2.45^2 = 6.0025$, it follows $\sqrt{6} < 2.45$. So, it is enough to show that

$$\left(\frac{6}{5}\right)^3 > \left(\frac{5}{4}\right)^{2.45}$$

which is true if and only if

$$3^{60} \cdot 2^{267} > 10^{109}.$$

Using abacus magic, we compute

$$\begin{aligned} 2^{267} &= \frac{1}{8} \cdot 1024^{27} = \frac{1}{8} \cdot (1048576 \cdot 1024)^9 \\ &= \frac{1}{8} \cdot (1073741824)^9 \\ &> \frac{1}{8} \cdot (10737418)^9 \cdot 10^{18} \\ &> \frac{1}{8} \cdot (115292145300000 \cdot 10737418)^3 \cdot 10^{18} \\ &> \frac{1}{8} \cdot (1237939956 \cdot 10^7)^3 \cdot 10^{33} \\ &= 618969978^3 \cdot 10^{54} \end{aligned}$$

The main calculations are shown here.

10737418	1152921453
..75161926	..8070450171
...32212254	...3458764359
....751619268070450171
.....429496724611685812
.....107374181152921453
.....858993449223371624
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115292145306724	12379399562028354

On the other hand, we have

$$3^{60} = 59049^6 = (3486784401)^3$$

This means that we need only show

$$(3486784401 \cdot 618969978)^3 > 10^{49} \iff 3486784401 \cdot 618969978 \geq \sqrt[3]{10} \cdot 10^{16}$$

In fact, we will show that

$$34867844 \cdot 618969978 \geq \sqrt[3]{10} \cdot 10^{14}.$$

Computing the first bits of the partial products, the left hand side is more than $2.1582 \cdot 10^{15}$.
On the other hand

$$\begin{aligned} 2.158 \cdot 2.158 &= 4.656964 \\ 4.656964 \cdot 2.15 &= 10.0124726 > 10. \end{aligned}$$

This completes the proof.

Remark. In <https://aops.com/community/p10229839> it is elegantly shown that for $a \geq 0$,

$$\left(\frac{2a+2}{2a+1}\right)^{\sqrt{a+1}} > \left(\frac{2a+1}{2a}\right)^{\sqrt{a}}.$$

The main idea is the following set of equations:

$$\begin{aligned} 0 &\leq \iint_{\substack{x,y \in [a,a+1]^2 \\ x \leq y}} \left(\frac{y-x}{2\sqrt{x}(x+y)^2} - \frac{y-x}{2\sqrt{y}(x+y)^2} \right) dy dx \\ &= \int_a^{a+1} \int_a^{a+1} \frac{y-x}{2\sqrt{x}(x+y)^2} dy dx \\ &= \sqrt{a+1}(\log(2a+2) - \log(2a+1)) - \sqrt{a}(\log(2a+1) - \log(2a)). \end{aligned}$$