Bulgaria 2018/3

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Twitch Solves ISL

Episode 69

Problem

Prove that

$$\left(\frac{6}{5}\right)^{\sqrt{3}} > \left(\frac{5}{4}\right)^{\sqrt{2}}.$$

Video

https://youtu.be/xEy1PKfyDaM

Solution

Because $2.45^2 = 6.0025$, it follows $\sqrt{6} < 2.45$. So, it is enough to show that

$$\left(\frac{6}{5}\right)^3 > \left(\frac{5}{4}\right)^{2.45}$$

which is true if and only if

$$3^{60} \cdot 2^{267} > 10^{109}.$$

Using abacus magic, we compute

$$\begin{aligned} 2^{267} &= \frac{1}{8} \cdot 1024^{27} = \frac{1}{8} \cdot (1048576 \cdot 1024)^9 \\ &= \frac{1}{8} \cdot (1073741824)^9 \\ &> \frac{1}{8} \cdot (10737418)^9 \cdot 10^{18} \\ &> \frac{1}{8} \cdot (115292145300000 \cdot 10737418)^3 \cdot 10^{18} \\ &> \frac{1}{8} \cdot \left(1237939956 \cdot 10^7\right)^3 \cdot 10^{33} \\ &= 618969978^3 \cdot 10^{54} \end{aligned}$$

The main calculations are shown here.

```
10737418                1152921453
  75161926                8070450171
   32212254                3458764359
    75161926                8070450171
     42949672                4611685812
      10737418                1152921453
       85899344                9223371624
---------------         -----------------
115292145306724         12379399562028354
```

On the other hand, we have

$$3^{60} = 59049^6 = (3486784401)^3$$

This means that we need only show

$$(3486784401 \cdot 618969978)^3 > 10^{49} \iff 3486784401 \cdot 618969978 \geq \sqrt[3]{10} \cdot 10^{16}$$

In fact, we will show that

$$34867844 \cdot 618969978 \geq \sqrt[3]{10} \cdot 10^{14}.$$

Computing the first bits of the partial products, the left hand side is more than $2.1582 \cdot 10^{15}$. On the other hand

$$\begin{aligned} 2.158 \cdot 2.158 &= 4.656964 \\ 4.656964 \cdot 2.15 &= 10.0124726 > 10. \end{aligned}$$

This completes the proof.