

# **USAMO 1996/1**

**Evan Chen**

TWITCH SOLVES ISL

Episode 68

## **Problem**

Prove that the average of the numbers  $n \sin n^\circ$  for  $n = 2, 4, 6, \dots, 180$  is  $\cot 1^\circ$ .

## **Video**

<https://youtu.be/7YQwWvVdLSg>

## **External Link**

<https://aops.com/community/p353049>

## Solution

Because

$$n \sin n^\circ + (180 - n) \sin(180^\circ - n^\circ) = 180 \sin n^\circ$$

So enough to show that

$$\sum_{n=0}^{89} \sin(2n)^\circ = \cot 1^\circ$$

Let  $\zeta = \cos 2^\circ + i \sin 2^\circ$  be a primitive root. Then

$$\begin{aligned} \sum_{n=0}^{89} \frac{\zeta^n - \zeta^{-n}}{2i} &= \frac{1}{2i} \left[ \frac{\zeta^{90} - 1}{\zeta - 1} - \frac{\zeta^{-90} - 1}{\zeta^{-1} - 1} \right] \\ &= \frac{1}{2i} \left[ \frac{-2}{\zeta - 1} - \frac{-2}{\zeta^{-1} - 1} \right] \\ &= \frac{1}{-i} \frac{\zeta^{-1} - \zeta}{(\zeta - 1)(\zeta^{-1} - 1)} = i \cdot \frac{\zeta + 1}{\zeta - 1}. \end{aligned}$$

Also,

$$\begin{aligned} \cot 1^\circ &= \frac{\cos 1^\circ}{\sin 1^\circ} = \frac{(\cos 1^\circ)^2}{\cos 1^\circ \sin 1^\circ} \\ &= \frac{\frac{\cos 2^\circ + 1}{2}}{\frac{\sin 2^\circ}{2}} = \frac{\frac{1}{2}(\zeta + \zeta^{-1}) + 1}{\frac{1}{2i}(\zeta - \zeta^{-1})} \\ &= i \cdot \frac{(\zeta + 1)^2}{\zeta^2 - 1} = i \cdot \frac{\zeta + 1}{\zeta - 1}. \end{aligned}$$

So we're done.