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Twitch Solves ISL

Episode 68

Problem

Prove that the average of the numbers $n \sin n^\circ$ for $n = 2, 4, 6, \dots, 180$ is $\cot 1^\circ$.

Video

https://youtu.be/7YQwWvVdLSg

Solution

Because

$$n \sin n^\circ + (180 - n)\sin(180^\circ - n^\circ) = 180 \sin n^\circ$$

So enough to show that

$$\sum_{n=0}^{89} \sin(2n)^\circ = \cot 1^\circ$$

Let $\zeta = \cos 2^\circ + i \sin 2^\circ$ be a primitive root. Then

$$\begin{aligned}\sum_{n=0}^{89} \frac{\zeta^n - \zeta^{-n}}{2i} &= \frac{1}{2i}\left[\frac{\zeta^{90}-1}{\zeta - 1} - \frac{\zeta^{-90}-1}{\zeta^{-1}-1}\right] \\ &= \frac{1}{2i}\left[\frac{-2}{\zeta-1} - \frac{-2}{\zeta^{-1}-1}\right] \\ &= \frac{1}{-i}\frac{\zeta^{-1}-\zeta}{(\zeta-1)(\zeta^{-1}-1)} = i \cdot \frac{\zeta+1}{\zeta-1}.\end{aligned}$$

Also,

$$\begin{aligned}\cot 1^\circ &= \frac{\cos 1^\circ}{\sin 1^\circ} = \frac{(\cos 1^\circ)^2}{\cos 1^\circ \sin 1^\circ} \\ &= \frac{\frac{\cos 2^\circ + 1}{2}}{\frac{\sin 2^\circ}{2}} = \frac{\frac{1}{2}(\zeta + \zeta^{-1}) + 1}{\frac{1}{2i}(\zeta - \zeta^{-1})} \\ &= i \cdot \frac{(\zeta+1)^2}{\zeta^2 - 1} = i \cdot \frac{\zeta + 1}{\zeta - 1}.\end{aligned}$$

So we're done.