Shortlist 2012 N1

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Twitch Solves ISL

Episode 68

Problem

Call admissible a set A of integers that has the following property: If $x, y \in A$ (possibly $x = y$) then $x^2 + kxy + y^2 \in A$ for every integer k. Determine all pairs m, n of nonzero integers such that the only admissible set containing both m and n is the set of all integers.

Video

https://youtu.be/yvQEumKoBXI

Solution

The answer is $\gcd(m,n) = 1$.

If $\gcd(m,n) > 1$, one can just let A be multiples of $\gcd(m,n)$.

On the other hand, suppose $\gcd(m,n) = 1$. Let $P(x,y,k)$ be the statement. Then:

- $P(m,m,k)$ and $P(n,n,k)$ show all multiples of m^2 and n^2 are in A.
- $P(am^2, bn^2, 2)$ gives

$$a^2 \cdot m^4 + 2ab \cdot m^2n^2 + b^2n^4 = (am^2 + bn^2)^2 \in A$$

  which shows every perfect square is in A. In particular $1 \in A$.
- Now $P(1,1,k)$ implies $A = \mathbb{Z}$.