Shortlist 2012 N1

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Twitch Solves ISL

Episode 68

Problem

Call admissible a set A of integers that has the following property: If $x, y \in A$ (possibly $x = y$) then $x^2 + kxy + y^2 \in A$ for every integer k. Determine all pairs m, n of nonzero integers such that the only admissible set containing both m and n is the set of all integers.

Video

https://youtu.be/yvQEumKoBXI

External Link

https://aops.com/community/p3160599

Solution

The answer is $\gcd(m,n)=1$.

If $\gcd(m,n)>1$, one can just let A be multiples of $\gcd(m,n)$.

On the other hand, suppose $\gcd(m,n)=1$. Let $P(x,y,k)$ be the statement. Then:

- $P(m,m,k)$ and $P(n,n,k)$ show all multiples of m^2 and n^2 are in A.
- $P(am^2,bn^2,2)$ gives

$$a^2\cdot m^4+2ab\cdot m^2n^2+b^2n^4=(am^2+bn^2)^2\in A$$

  which shows every perfect square is in A. In particular $1\in A$.
- Now $P(1,1,k)$ implies $A=\mathbb{Z}$.