# Shortlist 2012 N1 <br> Evan Chen 

Twitch Solves ISL
Episode 68

## Problem

Call admissible a set $A$ of integers that has the following property: If $x, y \in A$ (possibly $x=y$ ) then $x^{2}+k x y+y^{2} \in A$ for every integer $k$. Determine all pairs $m, n$ of nonzero integers such that the only admissible set containing both $m$ and $n$ is the set of all integers.

## Video

https://youtu.be/yvQEumKoBXI

## External Link

https://aops.com/community/p3160599

## Solution

The answer is $\operatorname{gcd}(m, n)=1$.
If $\operatorname{gcd}(m, n)>1$, one can just let $A$ be multiples of $\operatorname{gcd}(m, n)$.
On the other hand, suppose $\operatorname{gcd}(m, n)=1$. Let $P(x, y, k)$ be the statement. Then:

- $P(m, m, k)$ and $P(n, n, k)$ show all multiples of $m^{2}$ and $n^{2}$ are in $A$.
- $P\left(a m^{2}, b n^{2}, 2\right)$ gives

$$
a^{2} \cdot m^{4}+2 a b \cdot m^{2} n^{2}+b^{2} n^{4}=\left(a m^{2}+b n^{2}\right)^{2} \in A
$$

which shows every perfect square is in $A$. In particular $1 \in A$.

- Now $P(1,1, k)$ implies $A=\mathbb{Z}$.

