

Shortlist 1998 N1

Evan Chen

TWITCH SOLVES ISL

Episode 68

Problem

Determine all pairs (x, y) of positive integers such that $x^2y + x + y$ is divisible by $xy^2 + y + 7$.

Video

https://youtu.be/iswj_GVcbo8

External Link

<https://aops.com/community/p124428>

Solution

The relation implies

$$\begin{aligned} xy^2 + y + 7 &\mid x^2y + x + y \\ xy^2 + y + 7 &\mid x^2y^2 + xy + y^2 \\ xy^2 + y + 7 &\mid y^2 - 7x. \end{aligned}$$

As $-7(xy^2 + y + 7) < y^2 - 7x < xy^2 + y + 7$, we should be all set. Indeed, we have an integer k such that:

$$y^2 - 7x + k(xy^2 + y + 7) = 0 \iff x = \frac{y^2 + k(y + 7)}{7 - k \cdot y^2}.$$

The denominator is promised to be positive, so $k \leq 6$. Checking all the cases:

- $(x, y) = (7n^2, 7n)$ for $k = 0$, for any n (this is just $y^2/7 = x$)
- $(x, y) = (3/2, 1)$ for $k = 1$, no good
- $(x, y) = (13/3, 2)$ for $k = 1$, no good
- $(x, y) = (19/3, 1)$ for $k = 2$, no good
- $(x, y) = (25/4, 1)$ for $k = 3$, no good
- $(x, y) = (11, 1)$ for $k = 4$
- $(x, y) = (41/2, 1)$ for $k = 5$, no good
- $(x, y) = (49, 1)$ for $k = 6$.

Hence the answer $(7n^2, 7n)$, $(11, 1)$ and $(49, 1)$, easily seen to work.