Shortlist 1998 N1

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Twitch Solves ISL

Episode 68

Problem

Determine all pairs (x, y) of positive integers such that x^2y+x+y is divisible by xy^2+y+7.

Video

https://youtu.be/iswj_GVcbo8

External Link

https://aops.com/community/p124428

Solution

The relation implies

$$\begin{aligned} xy^2+y+7 &\mid x^2y+x+y \\ xy^2+y+7 &\mid x^2y^2+xy+y^2 \\ xy^2+y+7 &\mid y^2-7x. \end{aligned}$$

As $-7(xy^2+y+7) < y^2-7x < xy^2+y+7$, we should be all set. Indeed, we have an integer k such that:

$$y^2-7x+k(xy^2+y+7)=0 \iff x=\frac{y^2+k(y+7)}{7-k\cdot y^2}.$$

The denominator is promised to be positive, so $k \le 6$. Checking all the cases:

- $(x,y)=(7n^2,7n)$ for $k=0$, for any n (this is just $y^2/7=x$)
- $(x,y)=(3/2,1)$ for $k=1$, no good
- $(x,y)=(13/3,2)$ for $k=1$, no good
- $(x,y)=(19/3,1)$ for $k=2$, no good
- $(x,y)=(25/4,1)$ for $k=3$, no good
- $(x,y)=(11,1)$ for $k=4$
- $(x,y)=(41/2,1)$ for $k=5$, no good
- $(x,y)=(49,1)$ for $k=6$.

Hence the answer $(7n^2,7n)$, $(11,1)$ and $(49,1)$, easily seen to work.