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Twitch Solves ISL

Episode 68

Problem

Prove that for all non-negative numbers x, y, z satisfying $x + y + z = 1$, one has

$$1 \le \frac{x}{1-yz} + \frac{y}{1-zx} + \frac{z}{1-xy} \le \frac{9}{8}.$$

Video

https://youtu.be/hJ_r74sfBaE

Solution

Subtracting $x+y+z$ shows the inequality is equivalent to

$$0 \le \sum_{\text{cyc}} \frac{xyz}{1-yz} \le \frac{1}{8}.$$

The left-hand side is now clear since $\max(yz, zx, xy) < 1$, and equality holds if $xyz = 0$.

For the right-hand side, we wish to show.

$$\sum_{\text{cyc}} \frac{xyz}{1-yz} \le \frac{1}{8} \iff \sum_{\text{cyc}} \frac{1}{1-yz} \le \frac{1}{8xyz}$$

Let $f(t) \overset{\text{def}}{=} \frac{1}{1-\frac{xyz}{t}}$. It's concave, so by Jensen,

$$\sum_{\text{cyc}} \frac{1}{1-\frac{xyz}{x}} \le \frac{3}{1-\frac{xyz}{\frac{x+y+z}{3}}} \le \frac{3}{1-3xyz}.$$

Hence it suffices to show

$$\frac{3}{1-3xyz} \le \frac{1}{8xyz}$$

for $x+y+z=1$, which is clear because $xyz \le (1/3)^3$ by AM-GM.