Vietnam 2014 Training

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Twitch Solves ISL

Episode 67

Problem

Let ABC be a triangle inscribed circle (O), orthocenter H. Points E, F lie on (O) such that $EF \parallel BC$. Point D is midpoint of HE. The line passing though O and parallel to AF cuts AB at G. Prove that $DG \perp DC$.

Video

https://youtu.be/FxemXWXw92s

Solution

We present two solutions.

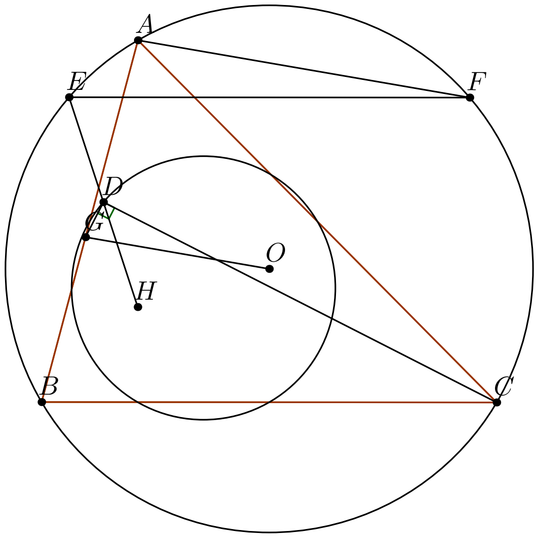
Complex numbers approach We use straight complex numbers with a, b, c, e as variables and with chord UV being the diameter of O parallel to $\overline{AF}$, so $G = \overline{UV} \cap \overline{AB}$. Then

$$\begin{aligned} f &= \frac{bc}{e} \\ d &= \frac{a+b+c+e}{2} \\ g &= \frac{ab(u+v) - uv(a+b)}{ab - uv} = \frac{-af(a+b)}{ab - af} = \frac{f(a+b)}{f-b} \\ \frac{g-d}{c-d} &= \frac{\frac{f(a+b)}{f-b} - \frac{a+b+c+e}{2}}{c - \frac{a+b+c+e}{2}} \\ &= -\frac{\frac{c(a+b)}{c-e} - \frac{a+b+c+e}{2}}{\frac{a+b+e-c}{2}} = \frac{1}{c-e} \cdot \frac{2c(a+b) - (c-e)(a+b+c+e)}{a+b+e-c} \\ &= \frac{1}{c-e} \cdot \frac{(c+e)(a+b) - (c-e)(c+e)}{a+b+e-c} = \frac{c+e}{c-e}. \end{aligned}$$

Moving points approach Fix ABC and animate E on the circumcircle. Then G varies on line AB projectively via the map

$$\begin{aligned} (ABC) \to (ABC) \to \ell_\infty \to AB \\ E \mapsto F \mapsto \infty \mapsto G. \end{aligned}$$

Also, D varies projectively on the nine-point circle.



Let $\infty_1 = \overline{DG} \cap \ell_\infty$ which has degree $2+1=3$. On the other hand, $\infty_2 = \overline{CD} \cap \ell_\infty$ has degree $2+0=2$. Hence, it suffices to verify the result for $3+2+1=6$ points, in order for the rotation of ∞_1 by 90° to coincide with ∞_2.

This can be done by letting D be the foot of the altitude and the midpoint of each of the three sides. (Proof should be written out in a contest, but here it is omitted.)