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Twitch Solves ISL

Episode 67

Problem

Let n be a positive integer, and let A_n be the set of all positive integers $a \leq n$ such that $n | a^n + 1$.

(a) Find all n such that $A_n \neq \emptyset$.

(b) Find all n such that $|A_n|$ is even and non-zero.

(c) Is there n such that $|A_n| = 130$?

Video

https://youtu.be/YBS-zrNl104

Solution

Part (a): The answer is odd n and even n with $\nu_2(n) \le 1$ and only $1 \bmod 4$ prime factors.

- For odd n, let $a = n - 1$ for a construction.
- For even n, the right-hand side is a sum of two squares, one of which is odd. So $\nu_2(n) \le 1$ and the 1 mod 4 condition are both needed. Conversely, we can choose n to be a square root of -1 modulo each prime power dividing n; this works.

Part (b): The answer is all even numbers in (a) other than $n = 2$. As for evenness:

- When n is even, pairing a and $-a$ implies $|A_n|$ even, except $n = 2$ (the only time $n \mid (n/2)^n + 1$).
- When n is odd, the number of solutions to $a^n \equiv -1 \pmod{p^e}$ for odd prime power p^e is odd: $a \equiv -1$ is a solution, $a \equiv 1$ is not, and any other solutions come in pairs $\{a, 1/a\}$.

Part (c): No. If such n existed, by (b) it is even.

- If n has more than one distinct odd prime factor, then $|A_n|$ is divisible by four, by Chinese remainder theorem.
- For $n = 2p^e$,
  $$a^{2p^e} \equiv -1 \pmod{p^e}$$
  We have a primitive root, of order $(p-1) \cdot p^{e-1}$. So there are $2p^{e-1}$ solutions.

Neither case coincides with 130, end proof.