Putnam 1979 B6

Evan Chen

Twitch Solves ISL

Episode 66

Problem

For $k = 1, 2, \dots, n$ let $z_k = x_k + iy_k$, where the x_k and y_k are real and $i = \sqrt{-1}$. Let r be the absolute value of the real part of

$$\pm\sqrt{z_1^2 + z_2^2 + \cdots + z_n^2}.$$

Prove that $r \le |x_1| + |x_2| + \cdots + |x_n|$.

Video

https://youtu.be/F1-jL2dSQJc

Solution

If the square root is $a + bi$, and $S = \sum x_k^2 \geq 0$ then

$$\begin{aligned}
(a^2 - b^2) + 2ab \cdot i &= \sum_k (x_k^2 - y_k^2) + \sum_k 2x_k y_k i \\
a^2 - b^2 &= \sum_k x_k^2 - y_k^2 \\
ab &= \sum_k x_k y_k \leq \sqrt{\sum_k x_k^2 \sum_k y_k^2} \\
&= \sqrt{\left(\sum_k x_k^2\right)\left(\sum_k x_k^2 - (a^2 - b^2)\right)} \\
a^2 b^2 &\leq S^2 - (a^2 - b^2) S \\
0 &\leq (S - a^2)(S + b^2).
\end{aligned}$$

This means

$$a^2 \leq \sum_k x_k^2 \implies a \leq \sqrt{\sum_k x_k^2} \leq \sum_k |x_k|.$$