

CAMO 2021/3

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TWITCH SOLVES ISL

Episode 66

Problem

Let ABC be an scalene triangle with circumcircle Γ and orthocenter H , and let K and M be the midpoints of \overline{AH} and \overline{BC} , respectively. Line AH intersects Γ again at T , and line KM intersects Γ at U and V . Lines TU and TV intersect lines AB and AC at X and Y , respectively, and point W lies on line KM such that $\overline{AW} \perp \overline{HM}$. If Z is the reflection of A over W , prove that X, Y, Z are collinear.

Video

<https://youtu.be/yRlq-SOP30E>

Analogous calculation gives

$$T = \left(\frac{-r[(a-r)^2 - b^2]}{(a-r)^2 + b^2}, \frac{-r \cdot 2b(a-r)}{(a-r)^2 + b^2} \right)$$

by replacing r with $-r$, and

$$R = \text{foot}(A, A'N) = \left(\frac{r[r^2 - (b/2)^2]}{r^2 + (b/2)^2}, \frac{r \cdot br}{r^2 + (b/2)^2} \right)$$

(obtained by setting $a \mapsto 0$, $b \mapsto b/2$ in the form for Q) We can now compute Z :

$$Z = \left(r + \frac{b}{Q_y}(Q_x - r), b \right) = \left(r + \frac{(a+r)^2 - b^2}{2(a+r)} - \frac{(a+r)^2 + b^2}{2(a+r)} \right) = \left(\frac{r(a+r) - b^2}{a+r}, b \right).$$

Finally,

$$\begin{aligned} \det(T, Z, R^*) &\asymp \det \begin{bmatrix} -r[(a-r)^2 - b^2] & -r \cdot 2(a-r) & (a-r)^2 + b^2 \\ r(a+r) - b^2 & (a+r) & a+r \\ r(r^2 - (b/2)^2) & r^2 & r^2 + (b/2)^2 \end{bmatrix} \\ &= \det \begin{bmatrix} -2r(a-r)^2 & -r \cdot 2(a-r) & (a-r)^2 + b^2 \\ -b^2 & a+r & a+r \\ -2r \cdot (b/2)^2 & r^2 & r^2 + (b/2)^2 \end{bmatrix} \\ &= \det \begin{bmatrix} -2r(a-r)^2 & -r \cdot 2(a-r) & a^2 + b^2 - r^2 \\ -b^2 & a+r & 0 \\ -2r \cdot (b/2)^2 & r^2 & (b/2)^2 \end{bmatrix} \\ &= (a^2 + b^2 - r^2) \left[-b^2 r^2 + \frac{1}{2} b^2 r(a+r) \right] \\ &\quad + (b/2)^2 \cdot (-2r(a-r)) \cdot [(a^2 - r^2) + b^2] \\ &= (a^2 + b^2 - r^2)(b/2)^2 [-4r^2 + 2r(a+r) - 2r(a-r)] = 0 \end{aligned}$$

as needed. □