# CAMO 2021/3 

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## Twitch Solves ISL

Episode 66

## Problem

Let $A B C$ be an scalene triangle with circumcircle $\Gamma$ and orthocenter $H$, and let $K$ and $M$ be the midpoints of $\overline{A H}$ and $\overline{B C}$, respectively. Line $A H$ intersects $\Gamma$ again at $T$, and line $K M$ intersects $\Gamma$ at $U$ and $V$. Lines $T U$ and $T V$ intersect lines $A B$ and $A C$ at $X$ and $Y$, respectively, and point $W$ lies on line $K M$ such that $\overline{A W} \perp \overline{H M}$. If $Z$ is the reflection of $A$ over $W$, prove that $X, Y, Z$ are collinear.

## Video

https://youtu.be/yRlq-S0P30E

## Solution

Construct $A^{\prime}$ the antipode, $Q=\overline{A^{\prime} H M} \cap \Gamma$, Call $N$ the midpoint of $\overline{U V}$.
If we define $Z^{\prime}=\overline{Q A} \cap \overline{X Y}, R^{\prime}=\overline{A Z^{\prime}} \cap \Gamma$, and $N^{\prime}=\overline{A^{\prime} R} \cap \overline{M K}$ then

$$
-1=(Q T ; B C) \stackrel{A}{=}\left(Z^{\prime}, \overline{A H T} \cap \overline{X Y Z^{\prime}} ; X, Y\right) \stackrel{A^{\prime}}{=}\left(R^{\prime} A ; U V\right)=\left(N^{\prime} \infty ; U V\right)
$$

where $\infty$ is the point at infinity along line $\overline{A A^{\prime}}\|\overline{M K}\| \overline{H Z}$. So $N^{\prime}=N$.


Now, define $Z$ instead as in the problem statement, and $R=\overline{A^{\prime} N} \cap \Gamma$. The problem is then proved upon showing the following claim (whence $Z^{\prime}=Z$ and $R=R^{\prime}$ ).

Claim. $T, R, Z$ collinear.
Proof. We use Cartesian coordinates. Set $O=(0,0), A^{\prime}=(-r, 0), A=(r, 0), H=(a, b)$, $N=(0, b / 2)$. To compute $Q$, we let $Q=(x, y)$ and solve the system

$$
\begin{aligned}
\frac{b}{a+r} & =\frac{y-b}{x-a}=\frac{r-x}{y} \\
x & =r-\frac{b}{a+r} \cdot y \Longrightarrow y=b+\frac{b}{a+r}(\underbrace{r-\frac{b}{a+r}}_{=x} y-a) \\
& \Longrightarrow\left(1+\frac{b^{2}}{(a+r)^{2}}\right) y=b+\frac{b(r-a)}{r+a} \\
\Longrightarrow Q & =\left(\frac{r\left[(a+r)^{2}-b^{2}\right]}{(a+r)^{2}+b^{2}}, \frac{r \cdot 2 b(a+r)}{(a+r)^{2}+b^{2}}\right)
\end{aligned}
$$

Analogous calculation gives

$$
T=\left(\frac{-r\left[(a-r)^{2}-b^{2}\right]}{(a-r)^{2}+b^{2}}, \frac{-r \cdot 2 b(a-r)}{(a-r)^{2}+b^{2}}\right)
$$

by replacing $r$ with $-r$, and

$$
R=\operatorname{foot}\left(A, A^{\prime} N\right)=\left(\frac{r\left[r^{2}-(b / 2)^{2}\right]}{r^{2}+(b / 2)^{2}}, \frac{r \cdot b r}{r^{2}+(b / 2)^{2}}\right)
$$

(obtained by setting $a \mapsto 0, b \mapsto b / 2$ in the form for $Q$ ) We can now compute $Z$ :
$Z=\left(r+\frac{b}{Q_{y}}\left(Q_{x}-r\right), b\right)=\left(r+\frac{(a+r)^{2}-b^{2}}{2(a+r)}-\frac{(a+r)^{2}+b^{2}}{2(a+r)}\right)=\left(\frac{r(a+r)-b^{2}}{a+r}, b\right)$.
Finally,

$$
\begin{aligned}
\operatorname{det}\left(T, Z, R^{*}\right) & \asymp \operatorname{det}\left[\begin{array}{ccc}
-r\left[(a-r)^{2}-b^{2}\right] & -r \cdot 2(a-r) & (a-r)^{2}+b^{2} \\
r(a+r)-b^{2} & (a+r) & a+r \\
r\left(r^{2}-(b / 2)^{2}\right) & r^{2} & r^{2}+(b / 2)^{2}
\end{array}\right] \\
= & \operatorname{det}\left[\begin{array}{ccc}
-2 r(a-r)^{2} & -r \cdot 2(a-r) & (a-r)^{2}+b^{2} \\
-b^{2} & a+r & a+r \\
-2 r \cdot(b / 2)^{2} & r^{2} & r^{2}+(b / 2)^{2}
\end{array}\right] \\
= & \operatorname{det}\left[\begin{array}{ccc}
-2 r(a-r)^{2} & -r \cdot 2(a-r) & a^{2}+b^{2}-r^{2} \\
-b^{2} & a+r & 0 \\
-2 r \cdot(b / 2)^{2} & r^{2} & (b / 2)^{2}
\end{array}\right] \\
= & \left(a^{2}+b^{2}-r^{2}\right)\left[-b^{2} r^{2}+\frac{1}{2} b^{2} r(a+r)\right] \\
& \quad+(b / 2)^{2} \cdot(-2 r(a-r)) \cdot\left[\left(a^{2}-r^{2}\right)+b^{2}\right] \\
= & \left(a^{2}+b^{2}-r^{2}\right)(b / 2)^{2}\left[-4 r^{2}+2 r(a+r)-2 r(a-r)\right]=0
\end{aligned}
$$

as needed.

