CAMO 2021/3

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Twitch Solves ISL

Episode 66

Problem

Let ABC be an scalene triangle with circumcircle Γ and orthocenter H, and let K and M be the midpoints of $\overline{AH}$ and $\overline{BC}$, respectively. Line AH intersects Γ again at T, and line KM intersects Γ at U and V. Lines TU and TV intersect lines AB and AC at X and Y, respectively, and point W lies on line KM such that $\overline{AW} \perp \overline{HM}$. If Z is the reflection of A over W, prove that X, Y, Z are collinear.

Video

https://youtu.be/yRlq-S0P3OE

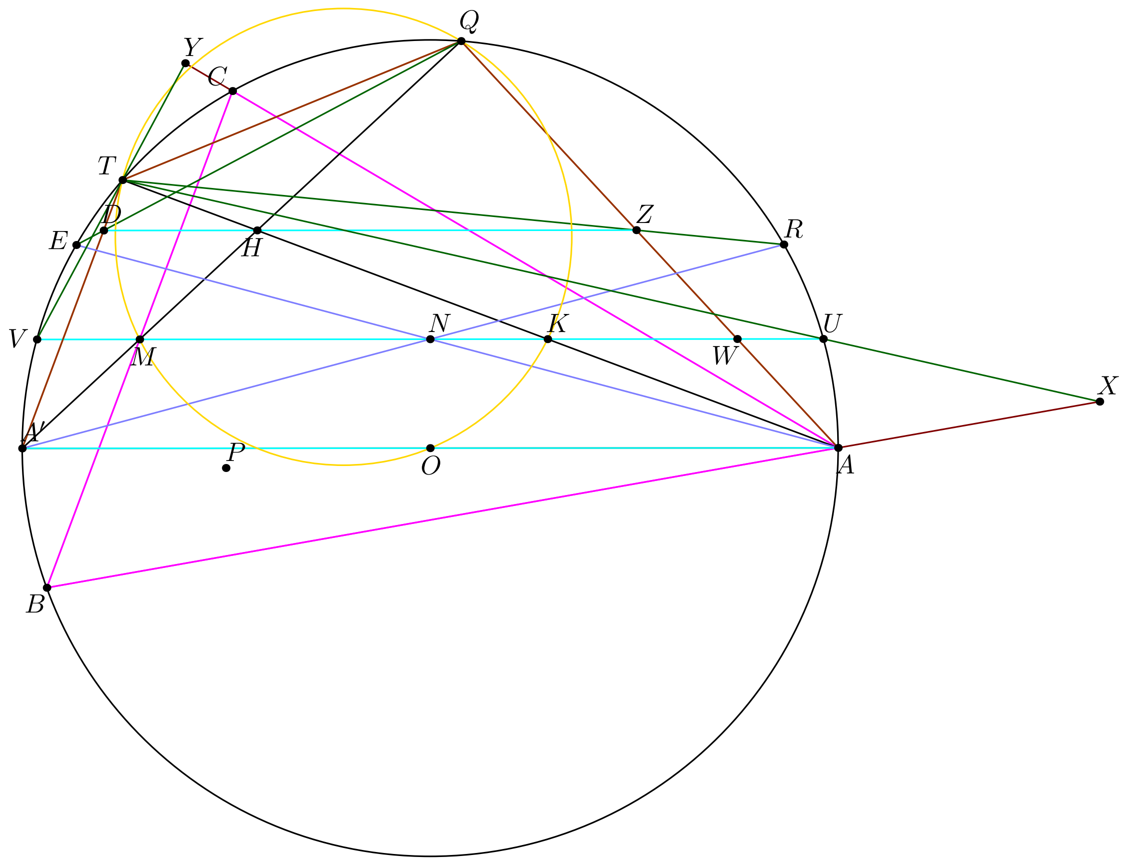
Solution

Construct A' the antipode, $Q = \overline{A'HM} \cap \Gamma$, Call N the midpoint of $\overline{UV}$.

If we define $Z' = \overline{QA} \cap \overline{XY}$, $R' = \overline{AZ'} \cap \Gamma$, and $N' = \overline{A'R} \cap \overline{MK}$ then

$$-1 = (QT;BC) \overset{A}{=} (Z', \overline{AHT} \cap \overline{XYZ'}; X, Y) \overset{A'}{=} (R'A; UV) = (N'\infty; UV)$$

where ∞ is the point at infinity along line $\overline{AA'} \parallel \overline{MK} \parallel \overline{HZ}$. So $N' = N$.



Now, define Z instead as in the problem statement, and $R = \overline{A'N} \cap \Gamma$. The problem is then proved upon showing the following claim (whence $Z' = Z$ and $R = R'$).

Claim. T, R, Z collinear.

Proof. We use Cartesian coordinates. Set $O = (0,0)$, $A' = (-r, 0)$, $A = (r, 0)$, $H = (a, b)$, $N = (0, b/2)$. To compute Q, we let $Q = (x, y)$ and solve the system

$$\frac{b}{a+r} = \frac{y-b}{x-a} = \frac{r-x}{y}$$

$$x = r - \frac{b}{a+r} \cdot y \implies y = b + \frac{b}{a+r}\Bigg(\underbrace{r - \frac{b}{a+r} y}_{=x} - a\Bigg)$$

$$\implies \left(1 + \frac{b^2}{(a+r)^2}\right) y = b + \frac{b(r-a)}{r+a}$$

$$\implies Q = \left(\frac{r[(a+r)^2 - b^2]}{(a+r)^2 + b^2}, \frac{r \cdot 2b(a+r)}{(a+r)^2 + b^2}\right).$$

Analogous calculation gives

$$T = \left(\frac{-r[(a-r)^2 - b^2]}{(a-r)^2 + b^2}, \frac{-r \cdot 2b(a-r)}{(a-r)^2 + b^2} \right)$$

by replacing r with $-r$, and

$$R = \operatorname{foot}(A, A'N) = \left(\frac{r[r^2 - (b/2)^2]}{r^2 + (b/2)^2}, \frac{r \cdot br}{r^2 + (b/2)^2} \right)$$

(obtained by setting $a \mapsto 0$, $b \mapsto b/2$ in the form for Q) We can now compute Z:

$$Z = \left(r + \frac{b}{Q_y}(Q_x - r), b \right) = \left(r + \frac{(a+r)^2 - b^2}{2(a+r)} - \frac{(a+r)^2 + b^2}{2(a+r)} \right) = \left(\frac{r(a+r) - b^2}{a+r}, b \right).$$

Finally,

$$\begin{aligned}
\det(T, Z, R^*) &\asymp \det \begin{bmatrix} -r[(a-r)^2 - b^2] & -r \cdot 2(a-r) & (a-r)^2 + b^2 \\ r(a+r) - b^2 & (a+r) & a+r \\ r(r^2 - (b/2)^2) & r^2 & r^2 + (b/2)^2 \end{bmatrix} \\
&= \det \begin{bmatrix} -2r(a-r)^2 & -r \cdot 2(a-r) & (a-r)^2 + b^2 \\ -b^2 & a+r & a+r \\ -2r \cdot (b/2)^2 & r^2 & r^2 + (b/2)^2 \end{bmatrix} \\
&= \det \begin{bmatrix} -2r(a-r)^2 & -r \cdot 2(a-r) & a^2 + b^2 - r^2 \\ -b^2 & a+r & 0 \\ -2r \cdot (b/2)^2 & r^2 & (b/2)^2 \end{bmatrix} \\
&= (a^2 + b^2 - r^2) \left[-b^2 r^2 + \frac{1}{2} b^2 r(a+r) \right] \\
&\quad + (b/2)^2 \cdot (-2r(a-r)) \cdot \left[(a^2 - r^2) + b^2 \right] \\
&= (a^2 + b^2 - r^2)(b/2)^2 \left[-4r^2 + 2r(a+r) - 2r(a-r) \right] = 0
\end{aligned}$$

as needed. □