

USAMO 2021/6

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TWITCH SOLVES ISL

Episode 65

Problem

Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X , Y , and Z be the midpoints of \overline{AD} , \overline{BE} , and \overline{CF} . Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.

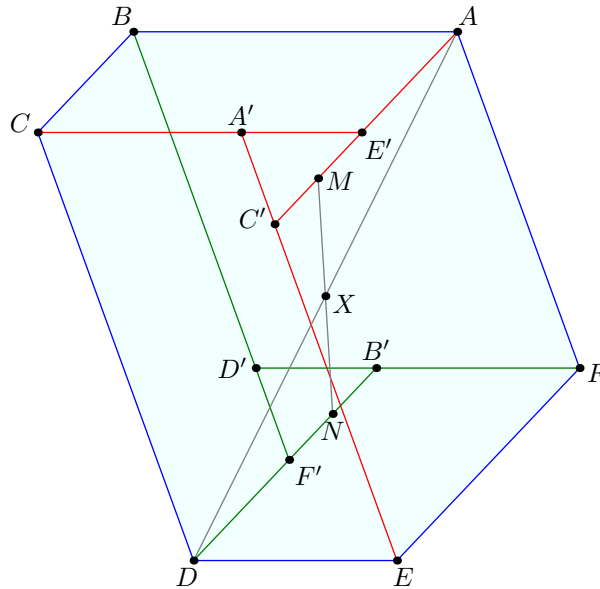
Video

<https://youtu.be/Nc4E8-QtjHk>

Solution

We present two solutions.

Parallelogram solution found by contestants Note that the following figure is intentionally *not* drawn to scale, to aid legibility. We construct parallelograms $ABCE'$, etc as shown. Note that this gives two congruent triangles $A'C'E'$ and $B'D'F'$. (Assuming that triangle XYZ is non-degenerate, the triangles $A'C'E'$ and $B'D'F'$ will also be non-degenerate.)



Claim. If $AB \cdot DE = BC \cdot EF = CD \cdot FA = k$, then the circumcenters of ACE and $A'C'E'$ coincide.

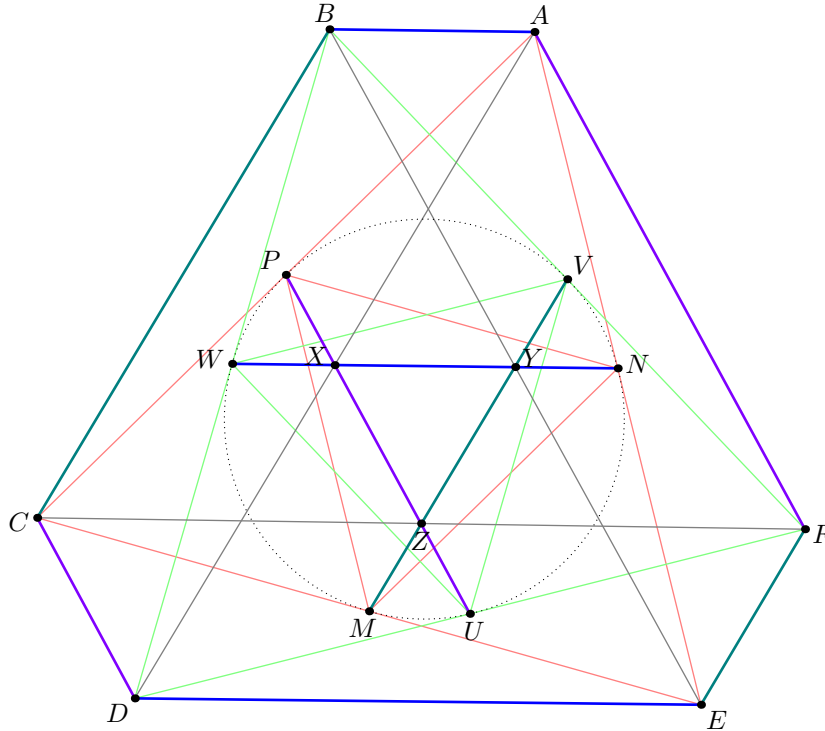
Proof. The power of A to $(A'C'E')$ is $AE' \cdot AC' = BC \cdot EF = k$; same for C and E . \square

Claim. Triangle XYZ is the vector average of the (congruent) medial triangles of triangles $A'C'E'$ and $B'D'F'$.

Proof. If M and N are the midpoints of $\overline{C'E'}$ and $\overline{B'F'}$, then X is the midpoint of \overline{MN} by vector calculation. \square

Hence the orthocenter of XYZ is the midpoint of the orthocenters of the medial triangles of $A'C'E'$ and $B'D'F'$ — that is, their circumcenters.

Author's solution Call MNP and UVW the medial triangles of ACE and BDF .



Claim. In trapezoid $ABDE$, the perpendicular bisector of \overline{XY} is the same as the perpendicular bisector of the midline \overline{WN} .

Proof. This is true for any trapezoid: because $WX = \frac{1}{2}AB = YN$. □

Claim. The points V, W, M, N are cyclic.

Proof. By power of a point from Y , since

$$WY \cdot YN = \frac{1}{2}DE \cdot \frac{1}{2}AB = \frac{1}{2}EF \cdot \frac{1}{2}BC = VY \cdot YM. \quad \square$$

Applying all the cyclic variations of the above two claims, it follows that all six points U, V, W, M, N, P are concyclic, and the center of that circle coincides with the circumcenter of $\triangle XYZ$.

Remark. It is also possible to implement ideas from the first solution here, by showing all six midpoints have equal power to (XYZ) .

Claim. The orthocenter of XYZ is the midpoint of the circumcenters of $\triangle ACE$ and $\triangle BDF$.

Proof. Apply complex numbers with the unit circle coinciding with the circumcircle of $NVPWMU$. Then

$$\begin{aligned} \text{orthocenter}(XYZ) &= x + y + z = \frac{a + b + c + d + e + f}{2} \\ \text{circumcenter}(ACE) &= \text{orthocenter}(MNP) \\ &= m + n + p = \frac{c + e}{2} + \frac{e + a}{2} + \frac{a + c}{2} = a + c + e \\ \text{circumcenter}(BDF) &= \text{orthocenter}(UVW) \\ &= u + v + w = \frac{d + f}{2} + \frac{f + b}{2} + \frac{b + d}{2} = b + d + f. \quad \square \end{aligned}$$