USAMO 2021/6

Evan Chen

Twitch Solves ISL

Episode 65

Problem

Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X, Y, and Z be the midpoints of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.

Video

https://youtu.be/9WNgDETH0lI

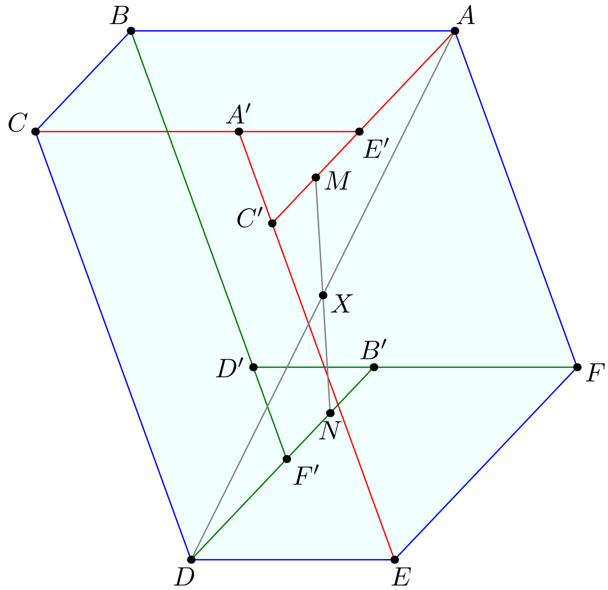
External Link

https://aops.com/community/p21498548

Solution

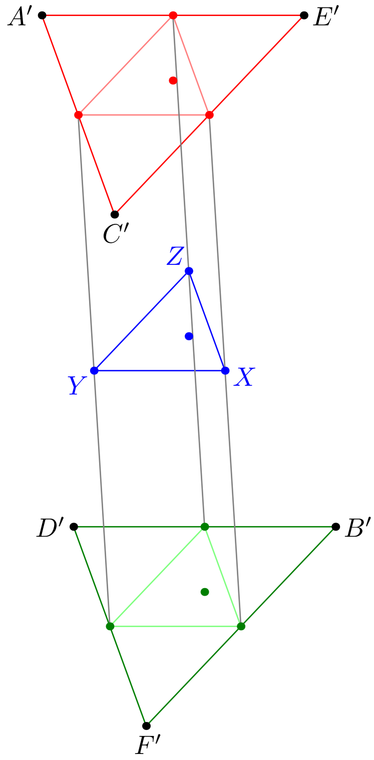
We present two solutions.

Parallelogram solution found by contestants. Note that the following figure is intentionally *not* drawn to scale, to aid legibility. We construct parallelograms $ABCE'$, etc as shown. Note that this gives two congruent triangles $A'C'E'$ and $B'D'F'$. (Assuming that triangle XYZ is non-degenerate, the triangles $A'C'E'$ and $B'D'F'$ will also be non-degenerate.)



Claim. If $AB \cdot DE = BC \cdot EF = CD \cdot FA = k$, then the circumcenters of ACE and $A'C'E'$ coincide.

Proof. The power of A to $(A'C'E')$ is $AE' \cdot AC' = BC \cdot EF = k$; same for C and E. $\square$



Claim. Triangle XYZ is the vector average of the (congruent) medial triangles of triangles $A'C'E'$ and $B'D'F'$.

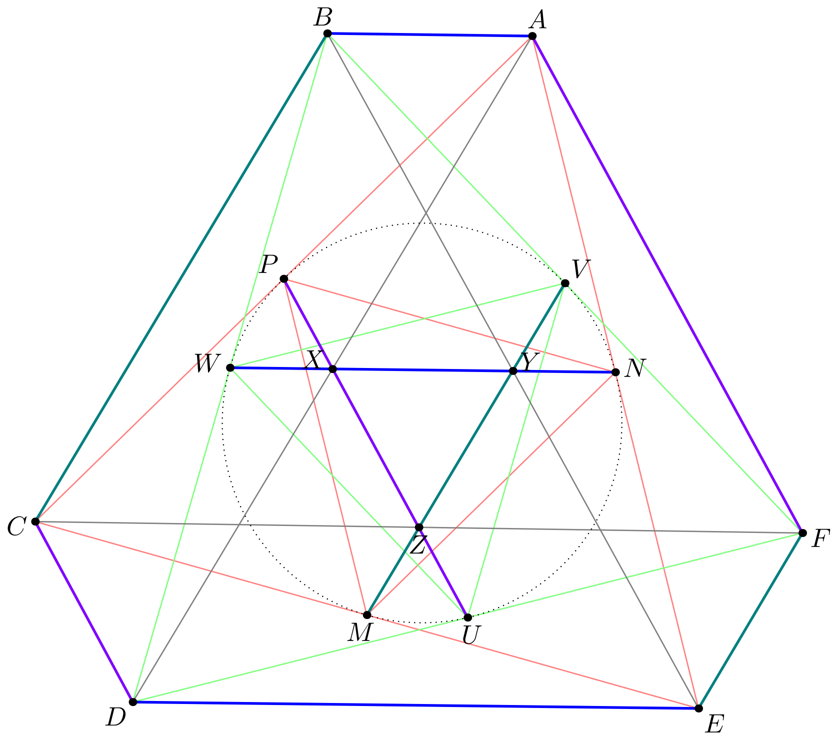
Proof. If M and N are the midpoints of $\overline{C'E'}$ and $\overline{B'F'}$, then X is the midpoint of $\overline{MN}$ by vector calculation:

$$\begin{aligned}\frac{\vec{M}+\vec{N}}{2} &= \frac{\frac{\vec{C}'+\vec{E}'}{2}+\frac{\vec{B}'+\vec{F}'}{2}}{2} \\ &= \frac{\vec{C}'+\vec{E}'+\vec{B}'+\vec{F}'}{4} \\ &= \frac{(\vec{A}+\vec{E}-\vec{F})+(\vec{C}+\vec{A}-\vec{B})+(\vec{D}+\vec{F}-\vec{E})+(\vec{B}+\vec{D}-\vec{C})}{4} \\ &= \frac{\vec{A}+\vec{D}}{2}=\vec{X}.\end{aligned}$$

□

Hence the orthocenter of XYZ is the midpoint of the orthocenters of the medial triangles of $A'C'E'$ and $B'D'F'$ — that is, their circumcenters.

Author's solution. Call MNP and UVW the medial triangles of ACE and BDF.



Claim. In trapezoid $ABDE$, the perpendicular bisector of $\overline{XY}$ is the same as the perpendicular bisector of the midline $\overline{WN}$.

Proof. This is true for any trapezoid: because $WX = \frac{1}{2}AB = YN$. $\square$

Claim. The points V, W, M, N are cyclic.

Proof. By power of a point from Y, since

$$WY \cdot YN = \frac{1}{2}DE \cdot \frac{1}{2}AB = \frac{1}{2}EF \cdot \frac{1}{2}BC = VY \cdot YM. \qquad \square$$

Applying all the cyclic variations of the above two claims, it follows that all six points U, V, W, M, N, P are concyclic, and the center of that circle coincides with the circumcenter of $\triangle XYZ$.

Remark. It is also possible to implement ideas from the first solution here, by showing all six midpoints have equal power to (XYZ).

Claim. The orthocenter of XYZ is the midpoint of the circumcenters of $\triangle ACE$ and $\triangle BDF$.

Proof. Apply complex numbers with the unit circle coinciding with the circumcircle of $NVPWMU$. Then

$$\begin{aligned}
\text{orthocenter}(XYZ) &= x + y + z = \frac{a+b+c+d+e+f}{2} \\
\text{circumcenter}(ACE) &= \text{orthocenter}(MNP) \\
&= m + n + p = \frac{c+e}{2} + \frac{e+a}{2} + \frac{a+c}{2} = a + c + e \\
\text{circumcenter}(BDF) &= \text{orthocenter}(UVW) \\
&= u + v + w = \frac{d+f}{2} + \frac{f+b}{2} + \frac{b+d}{2} = b + d + f. \qquad \square
\end{aligned}$$