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Twitch Solves ISL

Episode 65

Problem

Let $n \ge 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:

$$
\begin{aligned}
a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}, & a_2 &= a_1 + a_3, \\
a_3 &= \frac{1}{a_2} + \frac{1}{a_4}, & a_4 &= a_3 + a_5, \\
a_5 &= \frac{1}{a_4} + \frac{1}{a_6}, & a_6 &= a_5 + a_7, \\
&\vdots & &\vdots \\
a_{2n-1} &= \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}, & a_{2n} &= a_{2n-1} + a_1.
\end{aligned}
$$

Video

https://youtu.be/Nc4E8-QtjHk

Solution

The answer is that the only solution is $(1, 2, 1, 2, \dots, 1, 2)$ which works.

We will prove a_{2k} is a constant sequence, at which point the result is obvious.

First approach (Andrew Gu) Apparently, with indices modulo $2n$, we should have

$$a_{2k} = \frac{1}{a_{2k-2}} + \frac{2}{a_{2k}} + \frac{1}{a_{2k+2}}$$

for every index k (this eliminates all a_{odd}'s). Define

$$m = \min_k a_{2k} \qquad \text{and} \qquad M = \max_k a_{2k}.$$

Look at the indices i and j achieving m and M to respectively get

$$\begin{aligned} m &= \frac{2}{m} + \frac{1}{a_{2i-2}} + \frac{1}{a_{2i+2}} \ge \frac{2}{m} + \frac{1}{M} + \frac{1}{M} = \frac{2}{m} + \frac{2}{M} \\ M &= \frac{2}{M} + \frac{1}{a_{2j-2}} + \frac{1}{a_{2j+2}} \le \frac{2}{M} + \frac{1}{m} + \frac{1}{m} = \frac{2}{m} + \frac{2}{M}. \end{aligned}$$

Together this gives $m \ge M$, so $m = M$. That means a_{2i} is constant as i varies, solving the problem.

Second approach (author's solution) As before, we have

$$a_{2k} = \frac{1}{a_{2k-2}} + \frac{2}{a_{2k}} + \frac{1}{a_{2k+2}}$$

The proof proceeds in three steps.

- Define
  $$S = \sum_k a_{2k}, \quad \text{and} \quad T = \sum_k \frac{1}{a_{2k}}.$$
  Summing gives $S = 4T$. On the other hand, Cauchy-Schwarz says $S \cdot T \ge n^2$, so $T \ge \frac{1}{2}n$.
- On the other hand,
  $$1 = \frac{1}{a_{2k-2}a_{2k}} + \frac{2}{a_{2k}^2} + \frac{1}{a_{2k}a_{2k+2}}$$
  Sum this modified statement to obtain
  $$n = \sum_k \left(\frac{1}{a_{2k}} + \frac{1}{a_{2k+2}} \right)^2 \overset{\text{QM-AM}}{\ge} \frac{1}{n} \left(\sum_k \frac{1}{a_{2k}} + \frac{1}{a_{2k+2}} \right)^2 = \frac{1}{n} (2T)^2$$
  So $T \le \frac{1}{2}n$.
- Since $T \le \frac{1}{2}n$ and $T \ge \frac{1}{2}n$, we must have equality everywhere above. This means a_{2k} is a constant sequence.

Remark. The problem is likely intractable over $\mathbb{C}$, in the sense that one gets a high-degree polynomial which almost certainly has many complex roots. So it seems likely that most solutions must involve some sort of inequality, using the fact we are over $\mathbb{R}_{>0}$ instead.