USAMO 2021/4

Evan Chen

Twitch Solves ISL

Episode 65

Problem

A finite set S of positive integers has the property that, for each $s \in S$, and each positive integer divisor d of s, there exists a unique element $t \in S$ satisfying $\gcd(s,t) = d$. (The elements s and t could be equal.)

Given this information, find all possible values for the number of elements of S.

Video

https://youtu.be/Nc4E8-QtjHk

Solution

The answer is that $|S|$ must be a power of 2 (including 1), or $|S| = 0$ (a trivial case we do not discuss further).

Construction: For any nonnegative integer k, a construction for $|S| = 2^k$ is given by

$$S = \{(p_1 \text{ or } q_1) \times (p_2 \text{ or } q_2) \times \cdots \times (p_k \text{ or } q_k)\}$$

for $2k$ distinct primes $p_1, \ldots, p_k, q_1, \ldots, q_k$.

Converse: the main claim is as follows.

Claim. In any valid set S, for any prime p and $x \in S$, $\nu_p(x) \le 1$.

Proof. Assume for contradiction $e = \nu_p(x) \ge 2$.

- On the one hand, by taking x in the statement, we see $\frac{e}{e+1}$ of the elements of S are divisible by p.
- On the other hand, consider a $y \in S$ such that $\nu_p(y) = 1$ which must exist (say if $\gcd(x, y) = p$). Taking y in the statement, we see $\frac{1}{2}$ of the elements of S are divisible by p.

So $e = 1$, contradiction. $\square$

Now since $|S|$ equals the number of divisors of any element of S, we are done.