USAMO 2021/3

Evan Chen

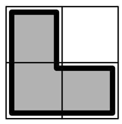
Twitch Solves ISL

Episode 65

Problem

Let $n \geq 2$ be an integer. An $n \times n$ board is initially empty. Each minute, you may perform one of three moves:

- If there is an L-shaped tromino region of three cells without stones on the board (see figure; rotations not allowed), you may place a stone in each of those cells.



- If all cells in a column have a stone, you may remove all stones from that column.
- If all cells in a row have a stone, you may remove all stones from that row.

For which n is it possible that, after some non-zero number of moves, the board has no stones?

Video

https://youtu.be/9WNgDETHOlI

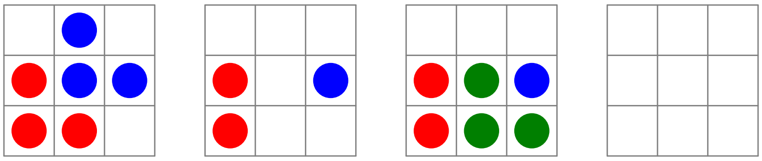
External Link

https://aops.com/community/p21498538

Solution

The answer is $3 \mid n$.

Construction: For $n = 3$, the construction is fairly straightforward, shown below.



This can be extended to any $3 \mid n$.

Polynomial-based proof of converse: Assume for contradiction $3 \nmid n$. We will show the task is impossible even if we allow stones to have real weights in our process. A valid elimination corresponds to a polynomial $P \in \mathbb{R}[x, y]$ such that

$$\begin{aligned} \deg_x P &\le n-2 \\ \deg_y P &\le n-2 \\ (1+x+y)P(x,y) &\in \left\langle 1+x+\cdots+x^{n-1},\ 1+y+\cdots+y^{n-1} \right\rangle . \end{aligned}$$

(Here $\langle \dots \rangle$ is an ideal of $\mathbb{R}[x, y]$.) In particular, if S is the set of nth roots of unity other than 1, we should have

$$(1+z_1+z_2)P(z_1, z_2) = 0$$

for any $z_1, z_2 \in S$. Since $3 \nmid n$, it follows that $1 + z_1 + z_2 \neq 0$ always.

So P vanishes on $S \times S$, a contradiction to the bounds on $\deg P$ (by, say, combinatorial nullstellensatz on any nonzero term).

Linear algebraic proof of converse (due to William Wang): Suppose there is a valid sequence of moves. Call r_j the number of operations clearing row j, indexing from bottom-to-top. The idea behind the solution is that we are going to calculate, for each cell, the number of times it is operated on entirely as a function of r_j. For example, a hypothetical illustration with $n = 6$ is partially drawn below, with the number in each cell denoting how many times it was the corner of an L.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 = r_3 & c_4 = r_5 - r_4 & c_5 = r_5 & 0 \\ \vdots & \vdots & r_2 + r_3 - r_5 & r_5 - r_3 & r_4 & 0 \\ \vdots & \vdots & r_1 + r_2 + r_3 - r_4 - r_5 & r_5 - r_2 & r_3 & 0 \\ \vdots & \vdots & r_1 + r_2 + r_4 - r_5 & r_5 - r_1 & r_2 & 0 \\ \vdots & \vdots & r_1 + r_4 - r_5 & r_5 & r_1 & 0 \end{bmatrix}$$

Let $a_{i,j}$ be the expression in (i, j). It will also be helpful to define c_i in the obvious way as well.

Claim. We have $c_n = r_n = 0$, $a_{n-1,j} = r_j$ and $a_{i,n-1} = c_i$.

Proof. The first statement follows since (n, n) may never obtain a stone. The equation $a_{n-1,j} = r_j$ follows as both equal the number of times that cell (n, j) obtains a stone. The final equation is similar. $\square$

Claim. For $1 \le i, j \le n-1$, the following recursion holds:

$$a_{i,j} + a_{i+1,j} + a_{i+1,j-1} = r_j + c_{i+1}.$$

Proof. Focus on cell $(i+1, j)$. The left-hand side counts the number of times that gains a stone while the right-hand side counts the number of times it loses a stone; they must be equal. $\square$

We can coerce the table above into matrix form now as follows. Define

$$K = \begin{bmatrix} -1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 0 \end{bmatrix}.$$

Then define a sequence of matrices M_i recursively by $M_{n-1} = \mathrm{id}$, and

$$M_i = \mathrm{id} + KM_{i+1} = \mathrm{id} + K + \dots + K^{n-1-i}.$$

The matrices are chosen so that, by construction,

$$\langle r_1, \dots, r_{n-1} \rangle \, M_i = \langle a_{i,1}, \dots, a_{i,n-1} \rangle$$

for $i = 1, 2, \dots, n-1$. On the other hand, we can extend the recursion one level deeper and obtain

$$\langle r_1, \dots, r_{n-1} \rangle \, M_0 = \langle 0, \dots, 0 \rangle \, .$$

However, the crux of the solution is the following.

Claim. The eigenvalues of K are exactly $-(1 + e^{\frac{2\pi i k}{n}})$ for $k = 1, 2, \dots, n-1$.

Proof. The matrix $-(K + \mathrm{id})$ is fairly known to have roots of unity as the coefficients. $\square$

However, we are told that apparently

$$0 = \det M_0 = \det \left(\mathrm{id} + K + K^2 + \dots + K^{n-1} \right)$$

which means $\det(K^n - \mathrm{id}) = 0$. This can only happen if K^n has eigenvalue 1, meaning that

$$\left[-(1 + \omega) \right]^n = 1$$

for ω some n^{th} root of unity, not necessarily primitive. This can only happen if $|1 + \omega| = 1$, ergo $3 \mid n$.