USAMO 2021/3

Evan Chen

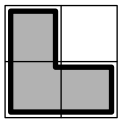
Twitch Solves ISL

Episode 65

Problem

Let $n \geq 2$ be an integer. An $n \times n$ board is initially empty. Each minute, you may perform one of three moves:

- If there is an L-shaped tromino region of three cells without stones on the board (see figure; rotations not allowed), you may place a stone in each of those cells.



- If all cells in a column have a stone, you may remove all stones from that column.
- If all cells in a row have a stone, you may remove all stones from that row.

For which n is it possible that, after some non-zero number of moves, the board has no stones?

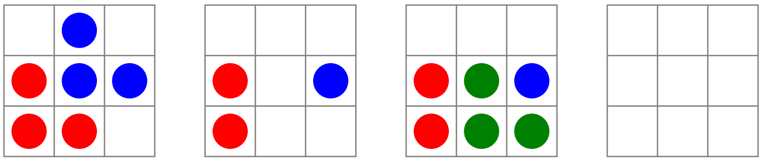
Video

https://youtu.be/Nc4E8-QtjHk

Solution

The answer is $3 \mid n$.

Construction: For $n = 3$, the construction is fairly straightforward, shown below.



This can be extended to any $3 \mid n$.

Converse: Assume for contradiction $3 \nmid n$. We will show the task is impossible even if we allow stones to have real weights in our process. A valid elimination corresponds to a polynomial $P \in \mathbb{R}[x, y]$ such that

$$\begin{aligned} \deg_x P &\le n - 2 \\ \deg_y P &\le n - 2 \\ (1 + x + y)P(x, y) &\in \left\langle 1 + x + \cdots + x^{n-1},\ 1 + y + \cdots + y^{n-1} \right\rangle . \end{aligned}$$

(Here $\langle \dots \rangle$ is an ideal of $\mathbb{R}[x, y]$.) In particular, if S is the set of nth roots of unity other than 1, we should have

$$(1 + z_1 + z_2)P(z_1, z_2) = 0$$

for any $z_1, z_2 \in S$. Since $3 \nmid n$, it follows that $1 + z_1 + z_2 \neq 0$ always.

So P vanishes on $S \times S$, a contradiction to the bounds on $\deg P$ (by, say, combinatorial nullstellensatz on any nonzero term).