USAMO 2021/1

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Twitch Solves ISL

Episode 65

Problem

Rectangles BCC_1B_2, CAA_1C_2, and ABB_1A_2 are erected outside an acute triangle ABC. Suppose that

$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^\circ.$$

Prove that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent.

Video

https://youtu.be/Nc4E8-QtjHk

Solution

The angle condition implies the circumcircles of the three rectangles concur at a single point P. Then $\measuredangle CPB_2 = \measuredangle CPA_1 = 90^\circ$, hence P lies on A_1B_2 etc., so we're done.

Remark. As one might guess from the two-sentence solution, the entire difficulty of the problem is getting the characterization of the concurrence point.