JMO 2021/4

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Twitch Solves ISL

Episode 65

Problem

Carina has three pins, labeled A, B, and C, respectively, located at the origin of the coordinate plane. In a *move*, Carina may move a pin to an adjacent lattice point at distance 1 away. What is the least number of moves that Carina can make in order for triangle ABC to have area 2021?

Video

https://youtu.be/9WNgDETHOlI

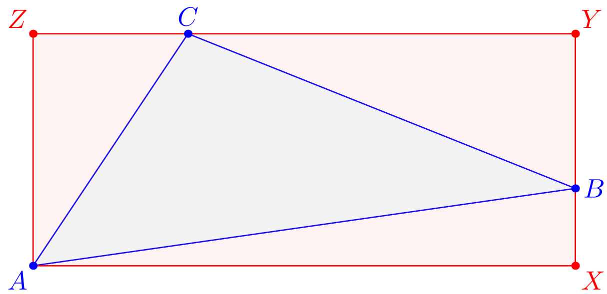
External Link

https://aops.com/community/p21498566

Solution

The answer is 128.

Define the **bounding box** of triangle ABC to be the smallest axis-parallel rectangle which contains all three of the vertices A, B, C.



Lemma. The area of a triangle ABC is at most half the area of the bounding box.

Proof. This can be proven by explicit calculation in coordinates. Nonetheless, we outline a geometric approach. By considering the smallest/largest x coordinate and the smallest/largest y coordinate, one can check that some vertex of the triangle must coincide with a corner of the bounding box (there are four "extreme" coordinates across the $3 \cdot 2 = 6$ coordinates of our three points).

So, suppose the bounding box is $AXYZ$. Imagine fixing C and varying B along the perimeter of the entire rectangle. The area is a linear function of B, so the maximal area should be achieved when B coincides with one of the vertices $\{A, X, Y, Z\}$. But obviously the area of $\triangle ABC$ is

- exactly 0 if $B = A$,
- at most half the bounding box if $B \in \{X, Z\}$ by one-half-base-height,
- at most half the bounding box if $B = Y$, since $\triangle ABC$ is contained inside either $\triangle AYZ$ or $\triangle AXZ$. □

We now proceed to the main part of the proof.

Claim. If n moves are made, the bounding box has area at most $(n/2)^2$. (In other words, a bounding box of area A requires at least $\left\lceil 2\sqrt{A} \right\rceil$ moves.)

Proof. The sum of the width and height of the bounding box increases by at most 1 each move, hence the width and height have sum at most n. So, by AM-GM, their product is at most $(n/2)^2$. □

This immediately implies $n \geq 128$, since the bounding box needs to have area at least $4042 > 63.5^2$.

On the other hand, if we start all the pins at the point $(3, 18)$ then we can reach the following three points in 128 moves:

$$\begin{aligned} A &= (0, 0) \\ B &= (64, 18) \\ C &= (3, 64) \end{aligned}$$

and indeed triangle ABC has area exactly 2021.

Remark. In fact, it can be shown that to obtain an area of $n/2$, the bounding–box bound of $\lceil 2\sqrt{n} \rceil$ moves is best possible, i.e. there will in fact exist a triangle with area $n/2$. However, since this was supposed to be a JMO4 problem, the committee made a choice to choose $n = 4042$ so that contestants only needed to give a single concrete triangle rather than a general construction for all integers n.