JMO 2021/3

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TWITCH SOLVES ISL

Episode 65

Problem

An equilateral triangle Δ of side length $L > 0$ is given. Suppose that n equilateral triangles with side length 1 and with non-overlapping interiors are drawn inside Δ, such that each unit equilateral triangle has sides parallel to Δ, but with opposite orientation. Prove that
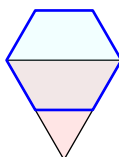
$$n \le \frac{2}{3} L^2.$$

Video

Solution

We present the approach of Andrew Gu. For each triangle, we draw a regular hexagon of side length $1/2$ as shown below.

Claim. All the hexagons are disjoint and lie inside Δ.

Proof. Annoying casework. $\square$

Since each hexagon has area $\frac{3\sqrt{3}}{8}$ and lies inside Δ, we conclude

$$\frac{3\sqrt{3}}{8} \cdot n \le \frac{\sqrt{3}}{4} L^2 \implies n \le \frac{2}{3} L^2.$$

Remark. The constant $\frac{2}{3}$ is sharp and cannot be improved, as the following tessellation shows: