

# JMO 2021/3

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TWITCH SOLVES ISL

Episode 65

## Problem

An equilateral triangle  $\Delta$  of side length  $L > 0$  is given. Suppose that  $n$  equilateral triangles with side length 1 and with non-overlapping interiors are drawn inside  $\Delta$ , such that each unit equilateral triangle has sides parallel to  $\Delta$ , but with opposite orientation. Prove that

$$n \leq \frac{2}{3}L^2.$$

## Video

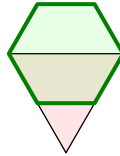
<https://youtu.be/9WNgDETH01I>

## External Link

<https://aops.com/community/p21499596>

## Solution

We present the approach of Andrew Gu. For each triangle, we draw a green regular hexagon of side length  $1/2$  as shown below.



**Claim.** All the hexagons are disjoint and lie inside  $\Delta$ .

*Proof.* Annoying casework. □

Since each hexagon has area  $\frac{3\sqrt{3}}{8}$  and lies inside  $\Delta$ , we conclude

$$\frac{3\sqrt{3}}{8} \cdot n \leq \frac{\sqrt{3}}{4} L^2 \implies n \leq \frac{2}{3} L^2.$$

**Remark.** The constant  $\frac{2}{3}$  is sharp and cannot be improved. The following tessellation shows how to achieve the  $\frac{2}{3}$  density. In the figure on the left, one of the green hexagons is drawn in for illustration. The version on the right has all the hexagons.

