JMO 2021/1

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Twitch Solves ISL

Episode 65

Problem

Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ which satisfy $f(a^2 + b^2) = f(a)f(b)$ and $f(a^2) = f(a)^2$ for all positive integers a and b.

Video

https://youtu.be/9WNgDETHOlI

External Link

https://aops.com/community/p21498724

Solution

The answer is $f \equiv 1$ only, which works. We prove it's the only one.

The bulk of the problem is:

Claim. If $f(a) = f(b) = 1$ and $a > b$, then $f(a^2 - b^2) = f(2ab) = 1$.

Proof. Write

$$\begin{aligned} 1 = f(a)f(b) = f(a^2 + b^2) &= \sqrt{f\left((a^2 + b^2)^2\right)} \\ &= \sqrt{f\left((a^2 - b^2)^2 + (2ab)^2\right)} \\ &= \sqrt{f(a^2 - b^2)f(2ab)}. \end{aligned}$$

□

By setting $a = b = 1$ in the given statement we get $f(1) = f(2) = 1$. Now a simple induction on n shows $f(n) = 1$:

- If $n = 2k$ take $(u, v) = (k, 1)$ hence $2uv = n$.
- If $n = 2k + 1$ take $(u, v) = (k + 1, k)$ hence $u^2 - v^2 = n$.