# Canada 2021/4 <br> Evan Chen <br> Twitch Solves ISL <br> Episode 65 

## Problem

A function $f: \mathbb{N} \rightarrow \mathbb{N}$ is called Canadian if it satisfies

$$
\operatorname{gcd}(f(f(x)), f(x+y))=\operatorname{gcd}(x, y)
$$

for all pairs of positive integers $x$ and $y$. Find all positive integers $m$ such that $f(m)=m$ for all Canadian functions $f$.

## Video

https://youtu.be/cnCr4S2tU04

## External Link

https://aops.com/community/p20902310

## Solution

This problem and solution were suggested by Eric Shen (CAN).
The answer is any integer which is not 1 or a prime power. The following functions show this, for any nonnegative integer $e$ (including $e=0$ ) and prime $p$ :

$$
f(x)= \begin{cases}p^{e+1} & x=p^{e} \\ p^{e} & x=p^{e+1} \\ x & \text { otherwise } .\end{cases}
$$

Here are two approaches.
Solution from Twitch Solves ISL. By letting $x=a$ and $x+y=b$, we get an addition-free statement:

$$
\operatorname{gcd}(f(f(a)), f(b))=\operatorname{gcd}(a, b) \quad \forall a<b
$$

In particular, if we fix a value of $b$, we see that $f(b)$ is divisible by any proper divisor of b.

Claim. We have $x \mid f(x)$ for any non prime power $x$.
Proof. Fix $b=x$ and vary $a$ across prime powers dividing $x$.
Claim. We have $f(x)=x$ for all $x$ which is not a prime power.
Proof. Since $x$ is not a prime power, we have $x \mid f(x)$, and in particular, $f(x)$ is not a prime power, so $x|f(x)| f(f(x))$. Now assume that $x<f(x)$; then

$$
f(f(x))=\operatorname{gcd}(f(f(x)), f(f(x)))=\operatorname{gcd}(x, f(x)) \mid x
$$

so $x=f(x)=f(f(x))$, contradiction.
Remark (Finding the construction). One can also deduce, though we don't need to, that $f(f(x))=x$ from here. Indeed fix $x$, and let $y=(x+f(f(x))+3)$ ! to conclude. Together with the fact that $p^{e-1} \mid f\left(p^{e}\right)$ for any prime $p$. (from $\left.(a, b)=\left(p^{e-1}, p^{e}\right)\right)$ this motivates the construction given at the start of the proof.

Approach by Kevin Min. We prove the following claim:
Claim. $f(f(x))=x$ for all $x$
Proof. Let $P(x, y)$ denote the given equation. For any prime $p$ and number $x$, let $v_{p}(x)=k$. Then note that if $p^{k+1} \mid f(f(x))$, then we cannot have $p^{k+1} \mid f(n)$ for any $n>f(x)$ or else $P(x, n-x)$ leads to a contradiction. But taking $P\left(p^{z}, p^{z}\right)$ for arbitrarily large $z$ we get $p^{k+1} \mid f(n)$ for arbitrarily large $n$, contradiction. Similarly, if $p^{k} \nmid f(f(x))$ then $P\left(x, p^{k}\right)$ gives a contradiction. Thus $v_{p}(f(f(x)))=v_{p}(x)$ for all primes p , so $f(f(x))=x$.

Then the given statement reduces to $\operatorname{gcd}(x, f(x+y))=\operatorname{gcd}(x, x+y)$. Let $z=x+y$ and let $Q(x, z)$ denote the given statement, where $z>x$. Then suppose $z$ is not a prime power, so for some prime $p$ let $v_{p}(z)=m, z=p^{m} \cdot c$. Then from $Q\left(p^{m}, z\right)$ we get $p^{m} \mid f(z)$, so $z \mid f(z)$. But then $f(z)$ isn't a prime power either, so $f(z) \mid f(f(z))$, and as $z=f(f(z))$ we must have $z=f(z)$.

