Canada 2021/4

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Twitch Solves ISL

Episode 65

Problem

A function $f\colon \mathbb{N} \to \mathbb{N}$ is called *Canadian* if it satisfies

$$\gcd\left(f(f(x)), f(x+y)\right) = \gcd(x, y)$$

for all pairs of positive integers x and y. Find all positive integers m such that $f(m) = m$ for all Canadian functions f.

Video

https://youtu.be/cnCr4S2tUO4

Solution

This problem and solution were suggested by Eric Shen (CAN).

The answer is any integer which is not 1 or a prime power. The following functions show this, for any nonnegative integer e (including $e = 0$) and prime p:

$$f(x) = \begin{cases} p^{e+1} & x = p^e \\ p^e & x = p^{e+1} \\ x & \text{otherwise.} \end{cases}$$

Here are two approaches.

Solution from Twitch Solves ISL By letting $x = a$ and $x+y = b$, we get an addition-free statement:

$$\gcd(f(f(a)), f(b)) = \gcd(a, b) \qquad \forall a < b.$$

In particular, if we fix a value of b, we see that $f(b)$ is divisible by any proper divisor of b.

Claim. We have $x \mid f(x)$ for any non prime power x.

Proof. Fix $b = x$ and vary a across prime powers dividing x. □

Claim. We have $f(x) = x$ for all x which is not a prime power.

Proof. Since x is not a prime power, we have $x \mid f(x)$, and in particular, $f(x)$ is not a prime power, so $x \mid f(x) \mid f(f(x))$. Now assume that $x < f(x)$; then

$$f(f(x)) = \gcd(f(f(x)), f(f(x))) = \gcd(x, f(x)) \mid x$$

so $x = f(x) = f(f(x))$, contradiction. □

Remark (Finding the construction)**.** One can also deduce, though we don't need to, that $f(f(x)) = x$ from here. Indeed fix x, and let $y = (x + f(f(x)) + 3)!$ to conclude. Together with the fact that $p^{e-1} \mid f(p^e)$ for any prime p. (from $(a, b) = (p^{e-1}, p^e)$) this motivates the construction given at the start of the proof.

Approach by Kevin Min We prove the following claim:

Claim. $f(f(x)) = x$ for all x

Proof. Let $P(x, y)$ denote the given equation. For any prime p and number x, let $v_p(x) = k$. Then note that if $p^{k+1} \mid f(f(x))$, then we cannot have $p^{k+1} \mid f(n)$ for any $n > f(x)$ or else $P(x, n - x)$ leads to a contradiction. But taking $P(p^z, p^z)$ for arbitrarily large z we get $p^{k+1} \mid f(n)$ for arbitrarily large n, contradiction. Similarly, if $p^k \nmid f(f(x))$ then $P(x, p^k)$ gives a contradiction. Thus $v_p(f(f(x))) = v_p(x)$ for all primes p, so $f(f(x)) = x$. □

Then the given statement reduces to $\gcd(x, f(x + y)) = \gcd(x, x + y)$. Let $z = x + y$ and let $Q(x, z)$ denote the given statement, where $z > x$. Then suppose z is not a prime power, so for some prime p let $v_p(z) = m, z = p^m \cdot c$. Then from $Q(p^m, z)$ we get $p^m \mid f(z)$, so $z \mid f(z)$. But then $f(z)$ isn't a prime power either, so $f(z) \mid f(f(z))$, and as $z = f(f(z))$ we must have $z = f(z)$.