# Canada 2021/4 Evan Chen

TWITCH SOLVES ISL

Episode 65

#### Problem

A function  $f \colon \mathbb{N} \to \mathbb{N}$  is called *Canadian* if it satisfies

 $\gcd\left(f(f(x)),f(x+y)\right)=\gcd(x,y)$ 

for all pairs of positive integers x and y. Find all positive integers m such that f(m) = m for all Canadian functions f.

### Video

https://youtu.be/cnCr4S2tU04

## **External Link**

https://aops.com/community/p20902310

#### Solution

This problem and solution were suggested by Eric Shen (CAN).

The answer is any integer which is not 1 or a prime power. The following functions show this, for any nonnegative integer e (including e = 0) and prime p:

$$f(x) = \begin{cases} p^{e+1} & x = p^e \\ p^e & x = p^{e+1} \\ x & \text{otherwise.} \end{cases}$$

Here are two approaches.

**Solution from Twitch Solves ISL.** By letting x = a and x + y = b, we get an addition-free statement:

$$gcd(f(f(a)), f(b)) = gcd(a, b) \quad \forall a < b.$$

In particular, if we fix a value of b, we see that f(b) is divisible by any proper divisor of *b*.

**Claim.** We have  $x \mid f(x)$  for any non prime power x.

*Proof.* Fix b = x and vary a across prime powers dividing x. 

**Claim.** We have f(x) = x for all x which is not a prime power.

*Proof.* Since x is not a prime power, we have  $x \mid f(x)$ , and in particular, f(x) is not a prime power, so  $x \mid f(x) \mid f(f(x))$ . Now assume that x < f(x); then

$$f(f(x)) = \gcd(f(f(x)), f(f(x))) = \gcd(x, f(x)) \mid x$$

so x = f(x) = f(f(x)), contradiction.

**Remark** (Finding the construction). One can also deduce, though we don't need to, that f(f(x)) = x from here. Indeed fix x, and let y = (x + f(f(x)) + 3)! to conclude. Together with the fact that  $p^{e-1} \mid f(p^e)$  for any prime p. (from  $(a, b) = (p^{e-1}, p^e)$ ) this motivates the construction given at the start of the proof.

**Approach by Kevin Min.** We prove the following claim:

**Claim.** f(f(x)) = x for all x

*Proof.* Let P(x,y) denote the given equation. For any prime p and number x, let  $v_p(x) = k$ . Then note that if  $p^{k+1} \mid f(f(x))$ , then we cannot have  $p^{k+1} \mid f(n)$  for any n > f(x) or else P(x, n - x) leads to a contradiction. But taking  $P(p^z, p^z)$  for arbitrarily large z we get  $p^{k+1} | f(n)$  for arbitrarily large n, contradiction. Similarly, if  $p^k \nmid f(f(x))$  then  $P(x, p^k)$  gives a contradiction. Thus  $v_p(f(f(x))) = v_p(x)$  for all primes p, so f(f(x)) = x.

Then the given statement reduces to gcd(x, f(x+y)) = gcd(x, x+y). Let z = x + yand let Q(x,z) denote the given statement, where z > x. Then suppose z is not a prime power, so for some prime p let  $v_p(z) = m, z = p^m \cdot c$ . Then from  $Q(p^m, z)$  we get  $p^m \mid f(z)$ , so  $z \mid f(z)$ . But then f(z) isn't a prime power either, so  $f(z) \mid f(f(z))$ , and as z = f(f(z)) we must have z = f(z).