# USAMO 1997/4 

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## Twitch Solves ISL

Episode 64

## Problem

To clip a convex $n$-gon means to choose a pair of consecutive sides $A B, B C$ and to replace them by the three segments $A M, M N$, and $N C$, where $M$ is the midpoint of $A B$ and $N$ is the midpoint of $B C$. In other words, one cuts off the triangle $M B N$ to obtain a convex $(n+1)$-gon. A regular hexagon $\mathcal{P}_{6}$ of area 1 is clipped to obtain a heptagon $\mathcal{P}_{7}$. Then $\mathcal{P}_{7}$ is clipped (in one of the seven possible ways) to obtain an octagon $\mathcal{P}_{8}$, and so on. Prove that no matter how the clippings are done, the area of $\mathcal{P}_{n}$ is greater than $\frac{1}{3}$, for all $n \geq 6$.

## Video

https://youtu.be/Nc4E8-QtjHk

## External Link

https://aops.com/community/p343875

## Solution

Call the original hexagon $A B C D E F$. We show the area common to triangles $A C E$ and $B D F$ is in every $\mathcal{P}_{n}$; this solves the problem since the area is $1 / 3$.

For every side of a clipped polygon, we define its foundation recursively as follows:

- $A B, B C, C D, D E, E F, F A$ are each their own foundation (we also call these original sides).
- When a new clipped edge is added, its foundation is the union of the foundations of the two edges it touches.

Hence, any foundations are nonempty subsets of original sides.
Claim. All foundations are in fact at most two-element sets of adjacent original sides.
Proof. It's immediate by induction that any two adjacent sides have at most two elements in the union of their foundations, and if there are two, they are two adjacent original sides.

Now, if a side has foundation contained in $\{A B, B C\}$, say, then the side should be contained within triangle $A B C$. Hence the side does not touch $A C$. This proves the problem.

