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Twitch Solves ISL

Episode 64

Problem

To clip a convex n-gon means to choose a pair of consecutive sides AB, BC and to replace them by the three segments AM, MN, and NC, where M is the midpoint of AB and N is the midpoint of BC. In other words, one cuts off the triangle MBN to obtain a convex $(n+1)$-gon. A regular hexagon $\mathcal{P}_6$ of area 1 is clipped to obtain a heptagon $\mathcal{P}_7$. Then $\mathcal{P}_7$ is clipped (in one of the seven possible ways) to obtain an octagon $\mathcal{P}_8$, and so on. Prove that no matter how the clippings are done, the area of $\mathcal{P}_n$ is greater than $\frac{1}{3}$, for all $n \ge 6$.

Video

https://youtu.be/Nc4E8-QtjHk

External Link

https://aops.com/community/p343875

Solution

Call the original hexagon $ABCDEF$. We show the area common to triangles ACE and BDF is in every $\mathcal{P}_n$; this solves the problem since the area is $1/3$.

For every side of a clipped polygon, we define its *foundation* recursively as follows:

- AB, BC, CD, DE, EF, FA are each their own foundation (we also call these *original sides*).
- When a new clipped edge is added, its foundation is the union of the foundations of the two edges it touches.

Hence, any foundations are nonempty subsets of original sides.

Claim. All foundations are in fact at most two-element sets of adjacent original sides.

Proof. It's immediate by induction that any two adjacent sides have at most two elements in the union of their foundations, and if there are two, they are two adjacent original sides. □

Now, if a side has foundation contained in $\{AB, BC\}$, say, then the side should be contained within triangle ABC. Hence the side does not touch AC. This proves the problem.