

# USAMO 1997/4

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TWITCH SOLVES ISL

Episode 64

## Problem

To clip a convex  $n$ -gon means to choose a pair of consecutive sides  $AB$ ,  $BC$  and to replace them by the three segments  $AM$ ,  $MN$ , and  $NC$ , where  $M$  is the midpoint of  $AB$  and  $N$  is the midpoint of  $BC$ . In other words, one cuts off the triangle  $MBN$  to obtain a convex  $(n + 1)$ -gon. A regular hexagon  $\mathcal{P}_6$  of area 1 is clipped to obtain a heptagon  $\mathcal{P}_7$ . Then  $\mathcal{P}_7$  is clipped (in one of the seven possible ways) to obtain an octagon  $\mathcal{P}_8$ , and so on. Prove that no matter how the clippings are done, the area of  $\mathcal{P}_n$  is greater than  $\frac{1}{3}$ , for all  $n \geq 6$ .

## Video

<https://youtu.be/Nc4E8-QtjHk>

## Solution

Call the original hexagon  $ABCDEF$ . We show the area common to triangles  $ACE$  and  $BDF$  is in every  $\mathcal{P}_n$ ; this solves the problem since the area is  $1/3$ .

For every side of a clipped polygon, we define its *foundation* recursively as follows:

- $AB, BC, CD, DE, EF, FA$  are each their own foundation (we also call these *original sides*).
- When a new clipped edge is added, its foundation is the union of the foundations of the two edges it touches.

Hence, any foundations are nonempty subsets of original sides.

**Claim.** All foundations are in fact at most two-element sets of adjacent original sides.

*Proof.* It's immediate by induction that any two adjacent sides have at most two elements in the union of their foundations, and if there are two, they are two adjacent original sides.  $\square$

Now, if a side has foundation contained in  $\{AB, BC\}$ , say, then the side should be contained within triangle  $ABC$ . Hence the side does not touch  $AC$ . This proves the problem.