Shortlist 2014 A2

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Twitch Solves ISL

Episode 64

Problem

Define the function $f\colon (0,1) \to (0,1)$ by

$$f(x) = \begin{cases} x + \frac{1}{2} & x < \frac{1}{2} \\ x^2 & x \geq \frac{1}{2}. \end{cases}$$

Let a and b be two real numbers such that $0 < a < b < 1$. We define the sequences a_n and b_n by $a_0 = a$, $b_0 = b$, and $a_n = f(a_{n-1})$, $b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer n such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0.$$

Video

https://youtu.be/LrqjMNxmzcw

Solution

Note that f has no fixed points. Assume for contradiction $a < b$ do not have this property. Call an index i *up* if $a_{i+1} > a_i$ and $b_{i+1} > b_i$, else a *down* index. It's clear there are infinitely many up indices.

Now let $d_n = |b_n - a_n|$.

Claim. We have

- If d_n is a down index, then $d_{n+1} = d_n$.
- If d_n is an up index, then $d_{n+1} \ge d_n + (b-a)^2$.

Proof. The first case is obvious. For the other case, note that

$$\begin{aligned} d_{n+1} &= \left|b_n^2 - a_n^2\right| = |b_n + a_n| \cdot |b_n - a_n| \\ &\ge \left(\frac{1}{2} + \left(\frac{1}{2} + d_n\right)\right) \cdot d_n \\ &\ge d_n + (b-a)^2. \end{aligned}$$

□

On the other hand, we obviously have $d_n < 1$ for all n. Since there are infinitely many up indices, the previous claim gives a contradiction.