Fary theorem

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Problem

Prove that every simple planar graph can be drawn with straight line segments as edges.

Solution

We'll only prove it for maximal planar graphs with three or more vertices: that is, graphs for which drawing any extra edge breaks planarity.

Claim. In a maximal planar graph, every edge is part of a cycle and every face is a triangle.

Proof. For the first part, There are no leaves, so every vertex has degree at least 2, hence part of a cycle. For the second part, note if there was a face with four or more edges, one can draw a chord. □

We prove by induction the following claim:

Claim. Every maximal planar graph G whose unbounded face is a triangle can be drawn with only straight line segments, in such a way that the faces and points inside them remain the same as in the original graph.

Proof. By induction. If G is a triangle, we are done.

Otherwise, using the bound $E \le 3n - 6$, there is a vertex v of degree at most 5 other than the vertices in the unbounded face. Mold $G - v$ with straight line segments. By art gallery problem, v can be connected to the vertices in the face containing v, as desired. □

Since any planar graph can be redrawn such that a triangle is the unbounded face, this implies the result.