Shortlist 2004 C2

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Twitch Solves ISL

Episode 62

Problem

Let n and k be positive integers. There are given n circles in the plane. Every two of them intersect at two distinct points, and all points of intersection they determine are pairwise distinct. No three circles have a point in common. Each intersection point must be colored with one of n distinct colors so that each color is used at least once and exactly k distinct colors occur on each circle. Find all values of $n \geq 2$ and k for which such a coloring is possible.

Video

External Link

Solution

Rephrasing:

> Given the graph $G = K_n$, we wish to color its edges with two of n colors (possibly the same color) such that every color is used, and every vertex v has exactly k colors used at it.

The answer is that:

- when $k = 2$, only $n \in \{2, 3\}$ are possible.

- when $k = 3$, any $n \geq k$ is permissible.

Let's prove this. In what follows, $S(v)$ will denote the set of colors at a vertex v.

For $k = 2$, the construction of $n = 2$ and $n = 3$ are routine:

- For $n = k = 2$, let $V(G) = \{v_1, v_2\}$. Then set $S(v_1) = S(v_2) = \{A, B\}$ and paint the edge with AB.

- For $n = 3$ and $k = 2$, let $V(G) = \{v_1, v_2, v_3\}$. Then set $S(v_1) = \{A, B\}$, $S(v_2) = \{B, C\}$, $S(v_3) = \{C, A\}$ and color v_1v_2 with BB, v_2v_3 with CC, v_3v_1 with AA.

Claim. For $k = 2$, we have $n \leq 3$.

Proof. Assume $n > 3$. Each $S(v)$ is a two-element set. So we get a natural graph Γ on the set of colors where the edges join two colors if they appear as some $S(v)$. This graph Γ has the property any two edges intersect. This means it is either a 3-cycle (hence $n = 3$) or $K_{1,n-1}$. For $n > 3$, the latter case cannot work, since every color must appear in at least two $S(v)$'s. $\square$

Claim. For $k \geq 3$ there is a construction for all $n \geq 3$.

Proof. If $n = k = 3$ let $V(G) = \{v_1, v_2, v_3\}$. Then set $S(v_1) = S(v_2) = S(v_3) = \{A, B, C\}$ and color v_1v_2 with AB, v_2v_3 with BC, v_3v_1 with CA.

Now suppose $n \geq 4$. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and let the n colors be C_1, C_2, ..., C_{n-1}, C_{special} (breaking symmetry). We will create a construction with

$$S(v_1) = \{C_{\text{special}}, C_1, C_2\}$$
$$S(v_2) = \{C_{\text{special}}, C_2, C_3\}$$
$$\vdots$$
$$S(v_{n-2}) = \{C_{\text{special}}, C_{n-2}, C_{n-1}\}$$
$$S(v_{n-1}) = \{C_{\text{special}}, C_{n-1}, C_1\}$$
$$S(v_n) = S(v_1) = \{C_{\text{special}}, C_1, C_2\}.$$

Assign colors to the following edges:

- Assign v_1v_2 colors $C_{\text{special}}C_2$.

- Assign v_2v_3 colors $C_{\text{special}}C_3$.

- ...

- Assign $v_{n-2}v_{n-1}$ colors $C_{\text{special}}C_{n-1}$.

- Assign $v_{n-1}v_n$ colors $C_{\text{special}}C_1$.

- Assign v_nv_1 colors C_1C_2.

Then assign $C_{\text{special}}C_{\text{special}}$ to all other edges. $\qquad\qquad\qquad\qquad\qquad\square$

Claim. Given a construction for (n,k) we can get a construction for $(n+1,k+1)$.

Proof. We do the following procedure.

- Create a new color C_{new} and add it to every $S(v)$.

- Add a new vertex v_{new}.

- Let C_1, ..., C_k be any old colors and let v_1, ..., v_k be vertices with $C_i \in S(v_i)$.

- Set $S(v_{\text{new}}) = \{C_{\text{new}}, C_1, C_2, \ldots, C_k\}$

- The edges $v_{\text{new}}v_i$ are colored C_iC_{new}.

- Any other edges are colored $C_{\text{new}}C_{\text{new}}$.

This completes the construction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark. I have to say I found most of the conditions in the problem rather unnatural to work with. For example, there is no real reason why the number of colors should be equal to the number of circles, and in fact the construction has to "work around this" by breaking the symmetry somewhat in the first main claim.