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Twitch Solves ISL

Episode 62

Problem

In the plane, let a, b be two closed broken lines (possibly self-intersecting), and K, L, M, N be four points. The vertices of a, b and the points K, L, M, N are in general position (i.e. no three of these points are collinear, and no three segments between them concur at an interior point). Each of segments KL and MN meets a at an even number of points, and each of segments LM and NK meets a at an odd number of points. Conversely, each of segments KL and MN meets b at an odd number of points, and each of segments LM and NK meets b at an even number of points. Prove that a and b intersect.

Video

Solution

Assume for contradiction this is not so.

Claim (Well-known)**.** The curve a encloses a region (meaning one can discuss being inside or outside a), and similarly for b.

Now:

- Since KN intersects a an odd number of times, exactly one of the two points is inside a. WLOG K is inside a and N is outside.

- Following through, M is outside, so L is inside.

- But then KL can't intersect b at all, contradiction.