Putnam 1972 B6

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Twitch Solves ISL

Episode 62

Problem

Let $n_1 < n_2 < n_3 < \cdots < n_k$ be a set of positive integers. Prove that the polynomial $1 + z^{n_1} + z^{n_2} + \cdots + z^{n_k}$ has no roots inside the circle $|z| < (\sqrt{5} - 1)/2$.

Video

https://youtu.be/iKfVuypCVos

Solution

Assume for contradiction z is a root with $|z| < \frac{\sqrt{5}-1}{2}$, and hence $|z| < 1$. Let $z = x + yi$ and $|z| = r$ for brevity.

Note that

$$-2\left(1 + z^{n_1} + \dots\right) + \underbrace{\left(z + z^2 + z^3 + \dots\right)}_{=\frac{z}{1-z}} = -2 \pm z \pm z^2 \pm \dots$$

Since z is a root of the polynomial, we have

$$\begin{aligned}
\frac{z}{1-z} &= -2 \pm z \pm z^2 \pm \dots \\
\iff \frac{2-z}{1-z} &= \pm z \pm z^2 \pm \dots \\
\implies \frac{|2-z|}{|1-z|} &\le |z| + |z|^2 + \dots = \\
\iff \frac{(2-z)(2-\overline{z})}{(1-z)(1-\overline{z})} &\le \frac{r^2}{(1-r)^2} \\
\iff \frac{4-4x+r^2}{1-2x+r^2} &\le \frac{r^2}{(1-r)^2} \\
\iff 2 + \frac{2-r^2}{1-2x+r^2} &\le \frac{r^2}{(1-r)^2} \\
\implies 2 + \frac{2-r^2}{1+2r+r^2} &\le \frac{r^2}{(1-r)^2} \\
\iff 2(1-r^2)^2 + (2-r^2)(1-r)^2 &\le r^2(1+r)^2 \\
\iff 0 &\le 4r^2 + 4r - 4
\end{aligned}$$

which implies $r \ge \frac{\sqrt{5}-1}{2}$, contradiction.

Remark. If $z = -\frac{\sqrt{5}-1}{2}$, then $z + z^3 + z^5 + \dots = -1$, so this is a sort of equality case for the problem.