CAMO 2021/2

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Twitch Solves ISL

Episode 62

Problem

Find all functions $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ satisfying

$$\big(f(x)+f(y)\big)^3 = xf(x) + yf(y) + 3f(xy)\big(f(x)+f(y)\big)$$

for all positive real numbers x and y.

Video

https://youtu.be/AlIvbT2FCI4

Solution

The answer is any function of the form

$$f(x) = \begin{cases} 0 & x \notin S \\ \sqrt{x} & x \in S \end{cases}$$

where S is a subset of $\mathbb{R}_{>0}$ which is closed under multiplication. The verification is left to the reader.

Let $k = f(1) \ge 0$, plug in $x = y = 1$ to get

$$(2k)^3 = 2k + 6k^2 \implies k \in \{0, 1\}.$$

Claim. If $k = 0$, then f is identically zero.

Proof. Plugging in $y = 1$ gives

$$f(x)^3 = xf(x) + 3f(x)^2 \implies f(x) \in \left\{0, \frac{3 + \sqrt{9 + 4x}}{2}\right\}.$$

We say t is bad if $f(t) \neq 0$. By plugging in $x = y = t$, we find that if t is bad then t^2 is bad and actually

$$8 \cdot \left(\frac{3 + \sqrt{9 + 4t}}{2}\right)^3 = 2t \cdot \frac{3 + \sqrt{9 + 4t}}{2} + 3 \cdot \frac{3 + \sqrt{9 + 4t^2}}{2} \cdot (3 + \sqrt{9 + 4t})$$

Dividing out:

$$2(3 + \sqrt{9 + 4t})^2 = 2t + 3(3 + \sqrt{9 + 4t^2}).$$

This has finitely many solutions in t though, while a single bad value of t (other than 1) should generate infinitely many. Contradiction. □

Otherwise, if $k = 1$, then plugging in $y = 1$ gives

$$(f(x) + 1)^3 = xf(x) + 1 + 3f(x)\left[f(x) + 1\right].$$

and hence

$$f(x)^3 = xf(x) \implies f(x) \in \left\{0, \sqrt{x}\right\}.$$

So, to finish, we only need to observe that if $f(x) = \sqrt{x}$ and $f(y) = \sqrt{y}$, then plugging in x and y shows $f(xy) = \sqrt{xy}$ as needed.