China TST 2021/1/3

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Twitch Solves ISL

Episode 60

Problem

Let n be a positive integer. Prove that for any integers a_1, a_2, $\ldots a_n$ at least $\lceil \frac{n(n-6)}{19} \rceil$ numbers from the set $\{1, 2, \ldots, \binom{n}{2}\}$ cannot be represented as $a_i - a_j$ where $1 \le i, j \le n$.

Video

https://youtu.be/VpupgdW1PBg

External Link

https://aops.com/community/p20980490

Solution

The idea is to consider the generating function

$$f(X) = \sum_i X^{a_i}.$$

Then, we may write

$$f(X)f(X^{-1}) - \sum_{-\binom{n}{2}}^{\binom{n}{2}} X^k = (n-1) + \sum_{k>0} c_k \left(X^k + X^{-k}\right)$$

for some constants c_k. It is easy to see that

- Every $c_k \geq -1$, and $c_k = -1$ exactly if k is "missed";
- $\sum c_k = 0$; and hence
- if t is the number of "missed" terms, then $\sum |c_k| = 2t$, and we want $t \geq \left\lceil \frac{n(n-6)}{19} \right\rceil$.

We now specialize to the choice

$$X = \omega = \exp\left(\frac{3\pi i}{n^2 - n + 1}\right).$$

On the one hand, the left-hand side is at most $(n-1) + 4t$, by the triangle inequality.

On the other hand, the left-hand side equals

$$|f(\omega)|^2 + \sum_{-\binom{n}{2}}^{\binom{n}{2}} \omega^k.$$

This is a real number, because the expression is self-conjugate.

We drop the first term, because it is nonnegative, and evaluate the second sum exactly:

$$\begin{aligned}
\sum_{-\binom{n}{2}}^{\binom{n}{2}} \omega^k &= -\omega^{-\binom{n}{2}} \cdot \frac{\omega^{n^2-n+1} - 1}{\omega - 1} = \frac{2}{\omega^{\binom{n}{2}+1} - \omega^{\binom{n}{2}}} = \frac{2}{\omega^{\binom{n}{2}+1} + \frac{1}{\omega^{\binom{n}{2}+1}}} \\
&= \sec\left(\frac{\binom{n}{2}+1}{n^2-n+1} \cdot 3\pi\right) = \frac{1}{\sin\left(\pi \cdot \frac{(n^2-n+1) - 3(n(n-1)+2)}{2n^2-2n+2}\right)} \\
&= \frac{1}{\sin\left(\pi \cdot \frac{-2n^2+2n-5}{2n^2-2n+2}\right)} = \frac{1}{\sin\left(\pi \cdot \left(\frac{-3}{2n^2-2n+2}\right) - \pi\right)} \\
&= \frac{1}{\sin\left(\pi \cdot \frac{3}{2n^2-2n+2}\right)} \geq \frac{2n^2-2n+2}{3\pi} \geq \frac{2n^2-2n+2}{19/2} \\
&= \frac{4n^2-4n+4}{19}.
\end{aligned}$$

Hence, it follows that

$$4t \geq \frac{4n^2-4n+4}{19} - (n-1) \geq \frac{4n^2-23n+23}{19} \geq \frac{4n^2-24n}{19}.$$

The problem is solved.

Remark. $6\pi < 19$.