CAMO 2021/6

Evan Chen

Twitch Solves ISL

Episode 60

Problem

If $a, b, c > 0$ and $a + b + c = 6$ prove that

$$\frac{a^2 - 4}{4a^2 - 9a + 6} + \frac{b^2 - 4}{4b^2 - 9b + 6} + \frac{c^2 - 4}{4c^2 - 9c + 6} \le 0.$$

Video

https://youtu.be/te7Lc344aIs

Solution

First, note that

$$\begin{aligned}\sum_{\text{cyc}} \left[\frac{3a^2-12}{4a^2-9a+6} + (2-a)\right] &= \sum_{\text{cyc}} \frac{(2-a)(4a^2-9a+6)+3a^2-12}{4a^2-9a+6} \\ &= \sum_{\text{cyc}} \frac{-4a^3+20a^2-24a}{4a^2-9a+6} \\ &= \sum_{\text{cyc}} \frac{-4a(a-2)(a-3)}{4a^2-9a+6}\end{aligned}$$

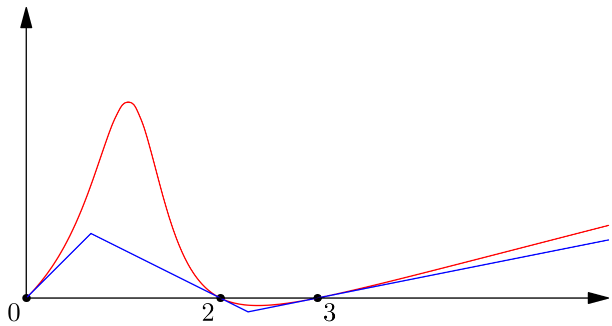
So, the inequality amounts to saying that

$$\sum_{\text{cyc}} \frac{a(a-2)(a-3)}{4a^2-9a+6} \geq 0 \qquad \forall a+b+c=6.$$

Let $f(x) = \frac{x(x-2)(x-3)}{4x^2-9x+6}$, and define the piecewise function $g\colon [0,6] \to \mathbb{R}$ by

$$g(x) = \begin{cases} x & x \in [0, 2/3] \\ -\frac{1}{2}(x-2) & x \in [2/3, 16/7] \\ \frac{1}{5}(x-3) & x \in [16/7, 6]. \end{cases}$$

The graph of f and g are drawn below in red and blue respectively.



Claim. We have $f(x) \geq g(x)$ for all $x \in [0,6]$.

Proof. Direct calculation. □

Hence, it suffices to show $g(a)+g(b)+g(c) \geq 0$ for all $a+b+c=6$. The proof is divided in cases:

- If $\min(a,b,c) \geq \frac{2}{3}$, then Jensen finishes.
- If $a \leq \frac{2}{3}$ and $b, c \geq \frac{2}{3}$, then by Jensen on the convex part of g, it is enough to show that $g(a) + 2g(3-a/2) \geq 0$, which is easy.
- If $a, b \leq \frac{2}{3}$, then $c \geq 3$, so all terms are nonnegative.

This finishes the proof.