

CAMO 2021/6

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TWITCH SOLVES ISL

Episode 60

Problem

If $a, b, c > 0$ and $a + b + c = 6$ prove that

$$\frac{a^2 - 4}{4a^2 - 9a + 6} + \frac{b^2 - 4}{4b^2 - 9b + 6} + \frac{c^2 - 4}{4c^2 - 9c + 6} \leq 0.$$

Video

<https://youtu.be/te7Lc344aIs>

Solution

First, note that

$$\begin{aligned} \sum_{\text{cyc}} \left[\frac{3a^2 - 12}{4a^2 - 9a + 6} + (2 - a) \right] &= \sum_{\text{cyc}} \frac{(2 - a)(4a^2 - 9a + 6) + 3a^2 - 12}{4a^2 - 9a + 6} \\ &= \sum_{\text{cyc}} \frac{-4a^3 + 20a^2 - 24a}{4a^2 - 9a + 6} \\ &= \sum_{\text{cyc}} \frac{-4a(a - 2)(a - 3)}{4a^2 - 9a + 6} \end{aligned}$$

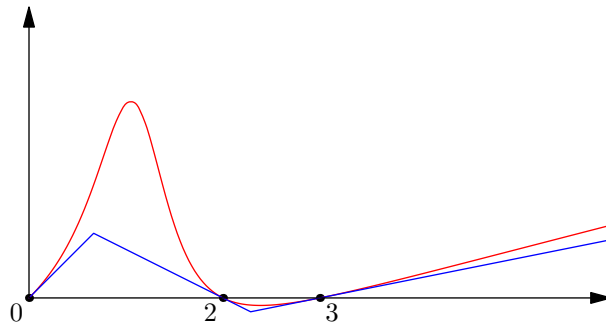
So, the inequality amounts to saying that

$$\sum_{\text{cyc}} \frac{a(a - 2)(a - 3)}{4a^2 - 9a + 6} \geq 0 \quad \forall a + b + c = 6.$$

Let $f(x) = \frac{x(x-2)(x-3)}{4x^2-9x+6}$, and define the piecewise function $g: [0, 6] \rightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} x & x \in [0, 2/3] \\ -\frac{1}{2}(x - 2) & x \in [2/3, 16/7] \\ \frac{1}{5}(x - 3) & x \in [16/7, 6]. \end{cases}$$

The graph of f and g are drawn below in red and blue respectively.



Claim. We have $f(x) \geq g(x)$ for all $x \in [0, 6]$.

Proof. Direct calculation. □

Hence, it suffices to show $g(a) + g(b) + g(c) \geq 0$ for all $a + b + c = 6$. The proof is divided in cases:

- If $\min(a, b, c) \geq \frac{2}{3}$, then Jensen finishes.
- If $a \leq \frac{2}{3}$ and $b, c \geq \frac{2}{3}$, then by Jensen on the convex part of g , it is enough to show that $g(a) + 2g(3 - a/2) \geq 0$, which is easy.
- If $a, b \leq \frac{2}{3}$, then $c \geq 3$, so all terms are nonnegative.

This finishes the proof.