Shortlist 1998 N8

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Twitch Solves ISL

Episode 59

Problem

Let $a_0, a_1, a_2, \ldots$ be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$, where i, j and k are not necessarily distinct. Determine a_{1998}.

Video

https://youtu.be/46z5jJ-rauc

Solution

It is clear by induction there is at most one sequence, since at any point a_i must be equal to the smallest integer not expressible using a_0 through a_{i-1}.

On the other hand, one can give an example of a sequence: take $\{a_i\}$ as a set to be the numbers that have only the digits 0 and 1 in their octal (base-8) representation.

Since $1998 = 2048 - 50 = 1984 + 14 = 11111001110_2$, it follows the answer is $11111001110_8 = 1{,}227{,}096{,}648$.