ELMO 2010/2

Evan Chen

Twitch Solves ISL

Episode 59

Problem

Let r and s be positive integers. Define $a_0 = 0$, $a_1 = 1$, and $a_n = ra_{n-1} + sa_{n-2}$ for $n \ge 2$. Let $f_n = a_1 a_2 \dots a_n$. Prove that $\frac{f_n}{f_k f_{n-k}}$ is an integer for all integers n and k such that $0 < k < n$.

Video

https://youtu.be/-tfVetEaVdo

Solution

It's equivalent to show that for integers k, m we have

$$a_1 \dots a_k \mid a_m \dots a_{m+k-1}. \qquad (\spadesuit)$$

From the theory of linear recurrences, we know there are algebraic integers α and β such that a_n has a closed form

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

(Namely, α and β are the roots of the polynomial $T^2 - rT - s = 0$. Note that since $s > 0$ it follows $\alpha \neq \beta$.)

Then, ($\spadesuit$) rewrites as

$$\frac{(\alpha^m - \beta^m)(\alpha^{m+1} - \beta^{m+1}) \dots (\alpha^{m+k-1} - \beta^{m+k-1})}{(\alpha - \beta)(\alpha^2 - \beta^2) \dots (\alpha^k - \beta^k)}.$$

Claim. We get a divisibility as polynomials in α and β.

Proof. Since this is homogeneous, it suffices to show for $\beta = 1$, $\alpha = X$:

$$\frac{(X^m - 1^m)(X^{m+1} - 1^{m+1}) \dots (X^{m+k-1} - 1^{m+k-1})}{(X - 1)(X^2 - 1^2) \dots (X^k - 1^k)}$$

If ζ is a primitive rth root of unity, it appears at least as many times on top as on bottom. $\square$

Hence, the number in ($\spadesuit$) is a rational algebraic integer, so it is an integer.