Shortlist 2000 N6

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Episode 58

Problem

Show that the set of positive integers that cannot be represented as a sum of distinct perfect squares is finite.

Video

https://youtu.be/2gpmGY2l2VU

Solution

Say a number is "good" if it satisfies the property. We start with the following base cases:

Claim. Every 1 mod 4 number from 37 to 133 is good, and in fact can be written using squares at most 121.

Proof. We only need to check numbers which have a 3 mod 4 prime factor with odd multiplicity, by Christmas theorem. This is easy:

$$\begin{aligned} 57 &= 6^2 + 4^2 + 2^2 + 1^2 \\ 69 &= 8^2 + 2^2 + 1^2 \\ 77 &= 8^2 + 3^2 + 2^2 \\ 93 &= 8^2 + 5^2 + 2^2 \\ 105 &= 10^2 + 2^2 + 1^2 \\ 129 &= 10^2 + 5^2 + 2^2 \\ 133 &= 8^2 + 7^2 + 4^2 + 2^2. \end{aligned}$$

□

Then, the following corollaries take place:

- Every 1 mod 4 number from 37 to 97 can be written with squares at most 81.
- By adding $10^2 = 100$, to the previous bullet, in tandem with the claim itself, every 1 mod 4 number from 37 to 197 can be written with squares in the set $\{1^2, 2^2, \dots, 11^2\}$.
- By adding $12^2 = 144$ to the previous bullet, every 1 mod 4 number from 37 to 331 can be written with squares in the set $\{1^2, 2^2, \dots, 11^2, 12^2\}$.
- By adding $14^2 = 196$ to the previous bullet, every 1 mod 4 number from 37 to 527 can be written with squares in the set $\{1^2, 2^2, \dots, 11^2, 12^2, 14^2\}$.
- ... and so on.

In this way, every sufficiently large 1 mod 4 number may be written as the sum of distinct squares from the set

$$\{1^2, 2^2, \dots, 11^2\} \cup \{12^2, 14^2, 16^2, 18^2 \dots\}.$$

Then a general sufficiently large number may be reduced to this case by subtracting 13^2, 15^2, 17^2 if necessary to reduce it to a 1 mod 4 number.