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Twitch Solves ISL

Episode 58

Problem

Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} (-1)^{\lfloor k(\sqrt{2}-1) \rfloor} \geq 0.$$

Video

https://youtu.be/IQWO75AEeyQ

Solution

For concreteness, we exhibit the following large table showing the first 17 terms:

$a_n = \lfloor (\sqrt{2}+2)n \rfloor$			3			6				10			13				17
$b_n = \lfloor \sqrt{2}n \rfloor$	1	2		4	5		7	8	9		11	12		14	15	16	
$c_n = \lfloor (\sqrt{2}-1)n \rfloor$	0	0		1	1		2	2	2		3	3		4	4	4	
$d_n = c_{n+1} - c_n$	0	1		0	1		0	0	1		0	1		0	0	1	

By Beatty's Theorem, the sequences a_n and b_n are disjoint and form a partition of $\mathbb{N}$. On the other hand, in the bottom sequence, the "consecutive runs" should all have length either 2 or 3; the ith block has length $a_i - a_{i-1} - 1$. (Here $a_0 = 0$ for convenience.)

Our task is to prove that $\sum_1^n (-1)^{c_n} \geq 0$. Grouping into blocks in the bottom ending with odd c_n, it is enough to show the following inequality for any k:

$$\begin{aligned}&(a_1 - a_0 - 1) + (a_3 - a_2 - 1) + \dots + (a_{2k-1} - a_{2k-2} - 1) \\ &\geq (a_2 - a_1 - 1) + (a_4 - a_3 - 1) + \dots + (a_{2k} - a_{2k-1} - 1).\end{aligned}$$

Using the fact that $a_k = 3k + c_k$, we can replace every a_k with c_k above. Then, rearranging gives the desired is equivalent to

$$\sum_{i=1}^{2k-1} (-1)^i \underbrace{(c_{i+1} - c_i)}_{=0 \text{ or } 1} \geq 0.$$

We recognize the inner term is just d_i. In fact, I claim that

$$\sum_{i=1}^{\ell} (-1)^i d_i \geq 0$$

for any integer ℓ.

Claim. If we read c_n from left to right, the indices for which c_n changes value correspond to the blocks in length 3 in d_n More explicitly, the ith block of d_i has length 3 if and only if $c_{i+1} \neq c_i$.

Proof. Imagine reading b_i from left to right. If b_i, b_{i+1} are adjacent (i.e. $b_{i+1} - b_i = 1$) then $a_{i+1} = a_i + 3$, and $a_{i+2} = a_{i+1} + 3$. So looking ahead, this gives two blocks of length 2 in the future. The proof is similar if the $b_{i+1} - b_i = 2$. $\square$

So suppose we take the first m blocks of d_i. Each individual block sums to ± 1 (because only the last bit is 1). Moreover, the sign of the ith block is $+1$ if and only if c_i is even.

Thus, through the long convoluted chain of reductions $n \to k \to \ell \to m$, we have the same inequality with a smaller input value (since $n > 2k \geq \ell > m$). Since the inequality is clearly true for base cases $n \leq 10$ (say), the proof is completed by strong induction.