# Putnam 2020 B6 <br> Evan Chen 

## Twitch Solves ISL

Episode 58

## Problem

Let $n$ be a positive integer. Prove that

$$
\sum_{k=1}^{n}(-1)^{\lfloor k(\sqrt{2}-1)\rfloor} \geq 0
$$

## Video

https://youtu.be/IQWO75AEeyQ

## External Link

https://aops.com/community/p20537381

## Solution

For concreteness, we exhibit the following large table showing the first 17 terms:

| $a_{n}=$ | $\lfloor(\sqrt{2}+2) n\rfloor$ | 3 |  |  | 6 |  |  | 10 |  | 13 |  | 17 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $b_{n}=$ | [ $2 n$ ] | 1 | 2 | 4 | 5 | 7 | 8 | 9 | 11 | 12 | 14 | 15 | 16 |  |
| $c_{n}=$ | ( $\sqrt{2}-1) n\rfloor$ | 0 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 | 4 | 4 |  |
| $d_{n}=$ | $c_{n+1}-c_{n}$ | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 |  |

By Beatty's Theorem, the sequences $a_{n}$ and $b_{n}$ are disjoint and form a partition of $\mathbb{N}$. On the other hand, in the bottom sequence, the "consecutive runs" should all have length either 2 or 3 ; the $i$ th block has length $a_{i}-a_{i-1}-1$. (Here $a_{0}=0$ for convenience.)

Our task is to prove that $\sum_{1}^{n}(-1)^{c_{n}} \geq 0$. Grouping into blocks in the bottom ending with odd $c_{n}$, it is enough to show the following inequality for any $k$ :

$$
\begin{aligned}
& \left(a_{1}-a_{0}-1\right)+\left(a_{3}-a_{2}-1\right)+\cdots+\left(a_{2 k-1}-a_{2 k-2}-1\right) \\
& \geq\left(a_{2}-a_{1}-1\right)+\left(a_{4}-a_{3}-1\right)+\cdots+\left(a_{2 k}-a_{2 k-1}-1\right) .
\end{aligned}
$$

Using the fact that $a_{k}=3 k+c_{k}$, we can replace every $a_{k}$ with $c_{k}$ above. Then, rearranging gives the desired is equivalent to

$$
\sum_{i=1}^{2 k-1}(-1)^{i}(\underbrace{c_{i+1}-c_{i}}_{=0 \text { or } 1}) \geq 0
$$

We recognize the inner term is just $d_{i}$. In fact, I claim that

$$
\sum_{i=1}^{\ell}(-1)^{i} d_{i} \geq 0
$$

for any integer $\ell$.
Claim. If we read $c_{n}$ from left to right, the indices for which $c_{n}$ changes value correspond to the blocks in length 3 in $d_{n}$. More explicitly, the $i$ th block of $d_{i}$ has length 3 if and only if $c_{i+1} \neq c_{i}$.

Proof. Imagine reading $b_{i}$ from left to right. If $b_{i}, b_{i+1}$ are adjacent (i.e. $b_{i+1}-b_{i}=1$ ) then $a_{i+1}=a_{i}+3$, and $a_{i+2}=a_{i+1}+3$. So looking ahead, this gives two blocks of length 2 in the future. The proof is similar if the $b_{i+1}-b_{i}=2$.

So suppose we take the first $m$ blocks of $d_{i}$. Each individual block sums to $\pm 1$ (because only the last bit is 1 ). Moreover, the sign of the $i$ th block is +1 if and only if $c_{i}$ is even.

Thus, through the long convoluted chain of reductions $n \rightarrow k \rightarrow \ell \rightarrow m$, we have the same inequality with a smaller input value (since $n>2 k \geq \ell>m$ ). Since the inequality is clearly true for base cases $n \leq 10$ (say), the proof is completed by strong induction.

