# CHMMC 2021/6 

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## Twitch Solves ISL

Episode 58

## Problem

Let $A B C$ be a triangle with circumcenter $O$. The interior bisector of $\angle B A C$ intersects $B C$ at $D$. Circle $\omega_{A}$ is tangent to segments $A B$ and $A C$ and internally tangent to the circumcircle at $P$. Let $E$ and $F$ be the points at which the $B$-excircle and $C$-excircle are tangent to $A C$ and $A B$. Suppose that lines $B E$ and $C F$ pass through a common point $N$ on the circumcircle of $A E F$.
(a) Prove that the circumcircle of $P D O$ passes through $N$.
(b) Suppose that $P D / B C=2 / 7$. Find, with proof, the value of $\cos \angle B A C$.

## Video

https://youtu.be/KZ6Gyc4an9M

## Solution

Let $V$ be the Bevan point, $G$ be the centroid, and $M_{b} M_{c}$ the $A$-midline. Let $M$ and $L$ be the arc midpoints.

Unconditionally, $V$ is the midpoint of $\overline{I O}, V$ is the antipode of $A$ on $(A E F)$, and $G$ lies on $\overline{I N}$ in a $1: 2$ ratio.


Claim. The point $I$ is the midpoint of $\overline{A M}$, and $N B=N C$.
Proof. We are given that $\measuredangle E N F=\measuredangle C A B=\measuredangle C N B$, so we may take a homothety of ratio $-\frac{1}{2}$ centered at $G$ which maps $N$ to $I, B$ to $M_{b}, C$ to $M_{c}$. The resulting circle is known to pass through $A$. Since $\left(A M_{b} M_{c}\right) \ni I$, then $I$ is the midpoint of $\overline{A M}$.

For the latter part, this follows from $I M_{b}=I M_{c}$.
Claim. Point $I$ lies on $(A E F V)$.
Proof. Since $I$ is the midpoint, it follows $\overline{V O I} \perp \overline{A I}$.
Claim. Point $D$ is the center of $I P M N$, and the midpoint of $\overline{I M}$.
Proof. Since $M A=2 M I=2 M B$, it follows by shooting lemma that $M D=\frac{1}{2} M I$. Also, we have $\measuredangle I N M=\measuredangle I N L=\measuredangle I A L=90^{\circ}$ and $\measuredangle M P I=90^{\circ}$, so we find that $N B M C$ is a rhombus, so $D$ is the midpoint of $\overline{I M}$.

Finally, to show prove part (a), we first note that $A D O N$ is cyclic from power of a point at $M$. Thus we compute

$$
\measuredangle P D N=2 \measuredangle P M N=2\left(90^{\circ}-\measuredangle M L P\right)=2 \measuredangle P L M=\measuredangle P O M=\measuredangle P O N
$$

For part (b), since $D P=\frac{1}{2} M I=\frac{1}{2} M B$, we have

$$
\begin{aligned}
\frac{4}{7} a & =B M=2 R \sin (A / 2)=\frac{a}{\sin A} \sin (A / 2)=\frac{a}{2 \cos (A / 2)} \\
\Longrightarrow \cos (A / 2) & =\frac{7}{8} \\
\cos A & =2 \cdot\left(\frac{7}{8}\right)^{2}-1=\frac{34}{64}=\frac{17}{32} .
\end{aligned}
$$

