CHMMC 2021/6

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Twitch Solves ISL

Episode 58

Problem

Let ABC be a triangle with circumcenter O. The interior bisector of $\angle BAC$ intersects BC at D. Circle ω_A is tangent to segments AB and AC and internally tangent to the circumcircle at P. Let E and F be the points at which the B-excircle and C-excircle are tangent to AC and AB. Suppose that lines BE and CF pass through a common point N on the circumcircle of AEF.

(a) Prove that the circumcircle of PDO passes through N.

(b) Suppose that $PD/BC = 2/7$. Find, with proof, the value of $\cos \angle BAC$.

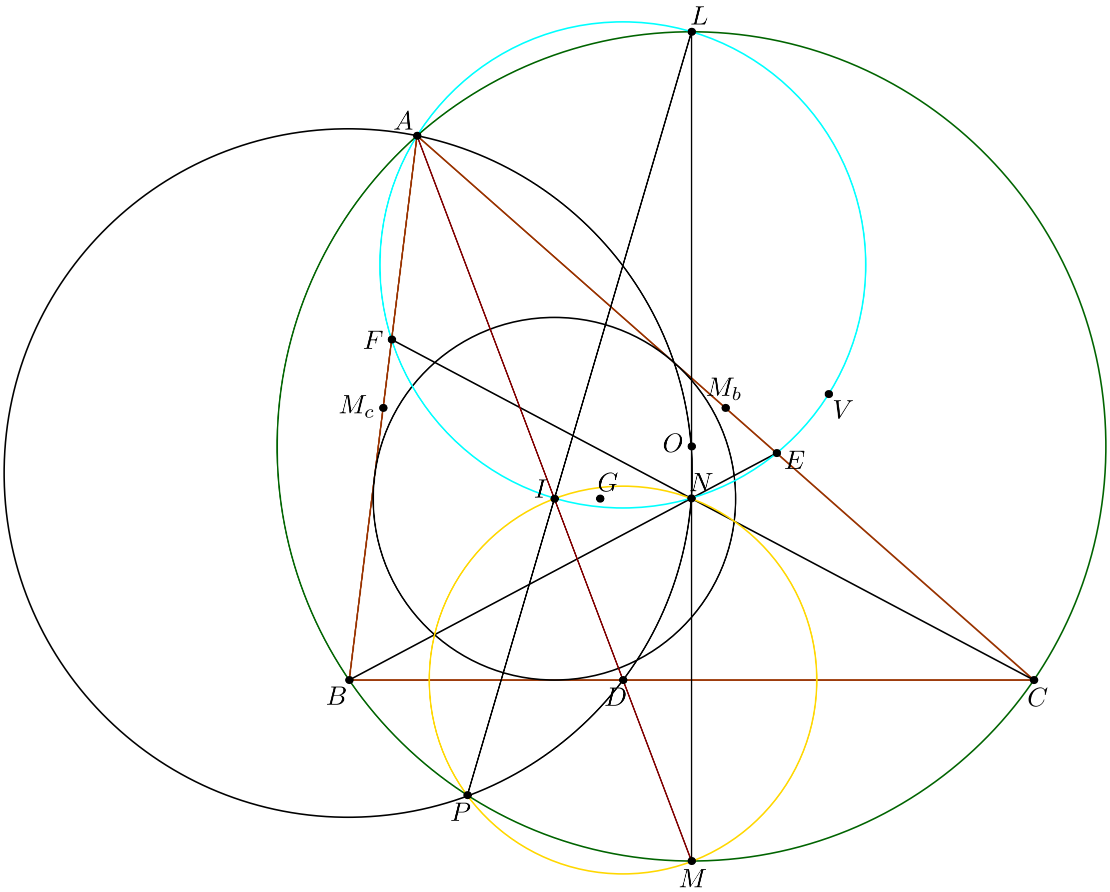
Video

https://youtu.be/KZ6Gyc4an9M

Solution

Let V be the Bevan point, G be the centroid, and M_bM_c the A-midline. Let M and L be the arc midpoints.

Unconditionally, V is the midpoint of $\overline{IO}$, V is the antipode of A on (AEF), and G lies on $\overline{IN}$ in a $1:2$ ratio.



Claim. The point I is the midpoint of $\overline{AM}$, and $NB = NC$.

Proof. We are given that $\measuredangle ENF = \measuredangle CAB = \measuredangle CNB$, so we may take a homothety of ratio $-\frac{1}{2}$ centered at G which maps N to I, B to M_b, C to M_c. The resulting circle is known to pass through A. Since $(AM_bM_c) \ni I$, then I is the midpoint of $\overline{AM}$.

For the latter part, this follows from $IM_b = IM_c$. □

Claim. Point I lies on $(AEFV)$.

Proof. Since I is the midpoint, it follows $\overline{VOI} \perp \overline{AI}$. □

Claim. Point D is the center of $IPMN$, and the midpoint of $\overline{IM}$.

Proof. Since $MA = 2MI = 2MB$, it follows by shooting lemma that $MD = \frac{1}{2}MI$. Also, we have $\measuredangle INM = \measuredangle INL = \measuredangle IAL = 90^\circ$ and $\measuredangle MPI = 90^\circ$, so we find that $NBMC$ is a rhombus, so D is the midpoint of $\overline{IM}$. □

Finally, to show prove part (a), we first note that $ADON$ is cyclic from power of a point at M. Thus we compute

$$\measuredangle PDN = 2\measuredangle PMN = 2(90^\circ - \measuredangle MLP) = 2\measuredangle PLM = \measuredangle POM = \measuredangle PON.$$

For part (b), since $DP = \frac{1}{2}MI = \frac{1}{2}MB$, we have

$$\begin{aligned} \frac{4}{7}a = BM = 2R\sin(A/2) &= \frac{a}{\sin A}\sin(A/2) = \frac{a}{2\cos(A/2)} \\ \implies \cos(A/2) &= \frac{7}{8} \\ \cos A &= 2\cdot\left(\frac{7}{8}\right)^2 - 1 = \frac{34}{64} = \frac{17}{32}. \end{aligned}$$