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Twitch Solves ISL

Episode 57

Problem

Arnaldo and Bernaldo play the following game: given a fixed finite set of positive integers A known by both players, Arnaldo picks a number $a \in A$ but doesn't tell it to anyone. Bernaldo then picks an arbitrary positive integer b (not necessarily in A). Then Arnaldo tells the number of divisors of ab. Show that Bernaldo can choose b in a way that he can find out the number a chosen by Arnaldo.

Video

https://youtu.be/TqUFlbW8cgE

Solution

Let p_0, p_1, p_2, $\dots$, p_k be the set of primes that appear anywhere in A. Define a huge integer N, such that

$$N > \prod_i \max_{a \in A} \nu_{p_i}(a)$$

for example (but anything big enough will do).

Then, we have Bernaldo select

$$b = p_0^{N-1} p_1^{N^2-1} p_2^{N^4-1} p_3^{N^8-1} \dots p_k^{N^{2^k}-1}.$$

Then, if Arnaldo had picked $a = p_0^{e_0} \dots p_k^{e_k}$, we would have

$$d(ab) = (N + e_0)(N^2 + e_1)(N^4 + e_2)(N^8 + e_3) \dots (N^{2^k} + e_k).$$

If we view this number in base N, then there is no re-grouping, because N is so large.

Then the $N^{2^{k+1}-1-2^i}$ digit would give e_i, so Bernaldo wins.

Remark. In general, Bernaldo can read *any* product from the digits: for example the N^5's place would give $e_1 e_3 e_4 e_5 \dots e_k$, or the N^{13} place would give $e_1 e_4 e_5 \dots e_k$. So only a few digits are necessary.