Besant 1895

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Twitch Solves ISL

Episode 57

Problem

Let k be a parabola with focus F. Let B and C be points on k, and suppose the tangents to k at B and C meet at a point A. Denote by O the circumcenter of $\triangle ABC$. Prove that $AF \perp FO$.

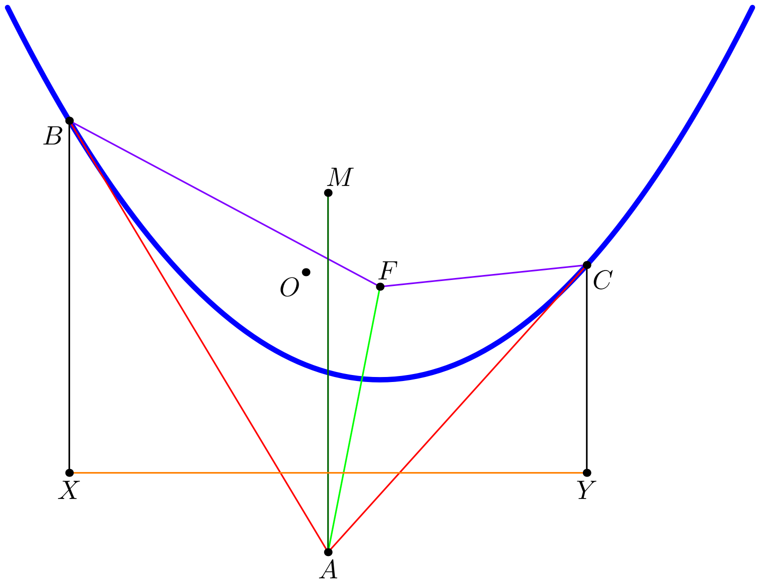
Video

https://youtu.be/2oMA0RpDBbA

Solution

We will show USAMO 2008 is equivalent to this problem, from which the reader can extract a synthetic proof (e.g. see David in https://aops.com/community/c5h202907p20773588)

Reflect F over AB and AC to obtain points X and Y. Then $BX = BF$, $CF = CY$. By the problem condition, $\measuredangle MAB = \measuredangle FBA = \measuredangle ABX$, so $BX \parallel MA$. Similarly, $CY \parallel MA$.



As $AX = AF = AY$, we have $XY \perp MA$. Therefore, we may draw a parabola through B and C, tangent to AB and AC, with focus F and with directrix coinciding with the line XY. Hence $\angle OFA = 90^\circ$ as needed.